The enclosure method for inverse obstacle scattering problems with dynamical data over a finite time interval: III. Sound-soft obstacle and bistatic data

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Abstract

This paper is concerned with an inverse obstacle problem which employs the dynamical scattering data of acoustic wave over a *finite time interval*. The unknown obstacle is assumed to be sound-soft one. The governing equation of the wave is given by the classical wave equation. The wave is generated by the initial data localized outside the obstacle and observed over a finite time interval at a place which is not necessary the same as the support of the initial data. The observed data are the so-called *bistatic data*. In this paper, an *enclosure method* which employs the bistatic data and is based on two main analytical formulae, is developed. The first one enables us to extract the maximum *spheroid* with focal points at the center of the support of the initial data and that of the observation points whose exterior encloses the unknown obstacle of general shape. The second one, under some technical assumption for the obstacle including convexity as an example, indicates the deviation of the geometry of the boundary of the obstacle and the maximum spheroid at the contact points. Several implications of those two formulae are also given. In particular, a constructive proof of a uniqueness of a *spherical* obstacle using the bistatic data is given.
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1 Introduction

In this paper, we consider an inverse obstacle scattering problem for a *sound-soft* obstacle with *dynamical data* over a *finite time interval*. The governing equation of the wave is the classical wave equation. The wave as the solution is generated by the initial data whose support is localized at the outside of the obstacle and observed over a finite time interval on a different position from the support of the initial data. The observed data are the

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so-called *bistatic data*. This is a simple mathematical model of the data collection process using an acoustic wave/electromagnetic wave such as, bistatic active *sonar*, *radar*, etc. See, e.g., [4] for the bistatic active sonar. The aim of this paper is to develop an enclosure method which employs the bistatic data.

Let us describe a mathematical formulation of the problem. Let D be a nonempty bounded open subset of $\mathbf{R}^3$ with C^2-boundary such that $\mathbf{R}^3 \setminus \overline{D}$ is connected. Let $0 < T < \infty$. Let $f \in L^2(\mathbf{R}^3)$ satisfy $\text{supp}\, f \cap \overline{D} = \emptyset$. Let $u = u_f(x,t)$ denote the weak solution of the following initial boundary value problem for the classical wave equation:

$$
\begin{array}{c}
\partial_t^2 u - \triangle u = 0 \text{ in } (\mathbf{R}^3 \setminus \overline{D})\times\,]0,\,T[,\\[2ex]
u(x,0) = 0 \text{ in } \mathbf{R}^3 \setminus \overline{D},\\[2ex]
\partial_t u(x,0) = f(x) \text{ in } \mathbf{R}^3 \setminus \overline{D},\\[2ex]
u = 0 \text{ on } \partial D\times\,]0,\,T[.
\end{array}
\tag{1.1}
$$

Here ν denotes the unit *outward normal* to D on ∂D. The boundary condition for u in (1.1) means that D is a *sound-soft* obstacle. In this paper, T is always fixed. Thus, for our purpose the weak solution over the bounded interval $]0,\,T[$ is appropriate. Since the notion of the weak solution for the wave equation is well established, we do not repeat the description here. Instead see [5] for the notion and also its use in [9, 11] for inverse obstacle scattering problems with dynamical data over a finite time interval.

In this paper, we consider the following problem.

Inverse Problem. Let B and B' be two *known* open balls centered at $p \in \mathbf{R}^3$ and $p' \in \mathbf{R}^3$ with radius η and η', respectively such that $\overline{B} \cap \overline{D} = \emptyset$ and $\overline{B}' \cap \overline{D} = \emptyset$. Let χ_B denote the characteristic function of B and set $f = \chi_B$. Assume that D is *unknown*. Extract information about the location and shape of D from the data $u_f(x,t)$ given at all $x \in B'$ and $t \in]0,\,T[$.

As far as the author knows, there is no result to this problem for general configulation of B and B'. This is the problem raised in [11] as an open problem related to the enclosure method itself. In particular, the problem contains the case when $\overline{B} \cap \overline{B'} = \emptyset$ which corresponds to the case when the emitter and receiver are placed on *different positions* at a *finite distance* from the obstacle. Strictly speaking, we should call the data in this case the bistatic data, however, we include also the case $\overline{B} \cap \overline{B'} \neq \emptyset$.

In this paper, we develop an enclosure method with bistatic data. In short, the enclosure method aims at extracting a domain that encloses an unknown discontinuity, such as cavities, cracks, inclusions or obstacles. The idea of the enclosure method goes back to [7], in which the original enclosure method was developed by considering an inverse boundary value problem governed by the Laplace equation. In [8], an idea for the application of the enclosure method to the dynamical data coming from the heat or wave equations has been introduced. Now we have many applications of this enclosure method to inverse boundary value problems governed by the heat equations in [14, 15, 10], visco-elastic system of equations [13] and inverse obstacle scattering problems governed by the wave equations in [9, 11, 12].

We establish two main analytical formulae. The first one enables us to extract the maximum *spheroid* with focal points at the center of the support of the initial data and

that of the observation points whose exterior encloses the unknown obstacle of general shape. The appearence of the exterior of a spheroid as an enclosing domain is *new* since previous enclosing domains are a half plane/space, sphere or its exterior, or cone. The formula shows us an effect of the bistatic data on the obtained information. See Theorem 1.1 below. The second one, under some technical assumption for the obstacle including convexity as an example, indicates the deviation of the geometry of the boundary of the obstacle and the maximum spheroid at the contact points. This is also new. See Theorem 1.3 below. And also we present several implications of those two formulae. In particular, we give a constructive proof of a uniqueness of a *spherical* obstacle using the bistatic data.

1.1 Extracting the first reflection distance and its implication

In this paper, given an arbitrary $h \in L^2(\mathbf{R}^3)$, we denote by v_h the unique weak solution $v \in H^1(\mathbf{R}^3)$ of

$$(\triangle - \tau^2)v + h(x) = 0 \text{ in } \mathbf{R}^3. \tag{1.2}$$

v_h has the expression

$$v_h(x) = v_h(x,\tau) = \frac{1}{4\pi}\int_{\mathbf{R}^3} \frac{e^{-\tau|x-y|}}{|x-y|} h(y)dy. \tag{1.3}$$

Define

$$w_f(x) = w_f(x,\tau) = \int_0^T e^{-\tau t} u_f(x,t)dt, \ x \in \mathbf{R}^3 \setminus \overline{D}, \ \tau > 0.$$

$w = w_f$ satisfies

$$\begin{aligned} &(\triangle - \tau^2)w + f(x) = e^{-\tau T} F_f(x,\tau) \text{ in } \mathbf{R}^3 \setminus \overline{D}, \\ &w = 0 \text{ on } \partial D, \end{aligned} \tag{1.4}$$

where

$$F_f(x,\tau) = \partial_t u_f(x,T) + \tau u_f(x,T), \ x \in \mathbf{R}^3 \setminus \overline{D}.$$

Since $F_f(x,\tau)$ is unknown, it seems that the existence of such term in (1.4) hides the information about an unknown obstacle. However, the use of the enclosure method presented below does not make it a problem at all and enables us to extract the information about the obstacle provided T is sufficiently large and fixed.

Let $\chi_{B'}$ denote the characteristic function of B' and set $g = \chi_{B'}$.

The results of this paper are concerned with the asymptotic behaviour of the *indicator function*:

$$\tau \longmapsto \int_{\mathbf{R}^3 \setminus \overline{D}} (f v_g - w_f g)dx = \int_B v_g dx - \int_{B'} w_f dx$$

For the description of the results we prepare some notation.

Define

$$\phi(x; y, y') = |y - x| + |x - y'|, \ (x, y, y') \in \mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{R}^3.$$

This is the length of the broken path connecting y to x and x to y' which plays the central role in this paper.

In this paper we denote the *convex hull* of the set $F \subset \mathbf{R}^3$ by $[F]$.

Theorem 1.1. *Let* $[\overline{B}\cup\overline{B}']\cap\partial D=\emptyset$ *and* T *satisfy*

$$T>\min_{x\in\partial D,\,y\in\partial B,\,y'\in\partial B'}\phi(x;y,y'). \tag{1.5}$$

Then, there exists a $\tau_0>0$ *such that, for all* $\tau\ge\tau_0$,

$$\int_{\mathbf{R}^3\setminus\overline{D}}(fv_g-w_fg)dx>0$$

and the formula

$$\lim_{\tau\longrightarrow\infty}\frac{1}{\tau}\log\int_{\mathbf{R}^3\setminus\overline{D}}(fv_g-w_fg)dx=-\min_{x\in\partial D,\,y\in\partial B,\,y'\in\partial B'}\phi(x;y,y') \tag{1.6}$$

is valid.

Note that

$$\min_{x\in\partial D,\,y\in\partial B,\,y'\in\partial B'}\phi(x;y,y')=\min_{x\in\partial D}\phi(x;p,p')-(\eta+\eta'). \tag{1.7}$$

See Appendix for the proof of (1.7). The quantity $\min_{x\in\partial D}\phi(x;p,p')$ coincides with the shortest length of the broken paths connecting p to a point q on ∂D and q to p', that is, the *first reflection distance* between p and q by D. (1.6) gives an extraction formula of $\min_{x\in\partial D}\phi(x;p,p')$ from $u_f(x,t)$ given at all $x\in B'$ and $t\in]0,\,T[$. Formula (1.6) gives the method of carrying out calculation processing of the waveform mathematically, and extracting the first reflection distance.

Define $[p,p']=\{sp+(1-s)p'\,|\,0\le s\le 1\}$. This is the straight line segment connecting the centers of B and B' and coincides with $[\{p,p'\}]$. Since both p and p' are in $\mathbf{R}^3\setminus\overline{D}$, $[p,p']\cap\overline{D}=\emptyset$ if and only if $[p,p']\cap\partial D=\emptyset$.

We know that

- $\min_{x\in\partial D}\phi(x;p,p')\ge|p-p'|$;
- if $[p,p']\cap\partial D=\emptyset$, then $\min_{x\in\partial D}\phi(x;p,p')>|p-p'|$.

Given $c>|p-p'|$ define

$$E_c(p,p')=\{x\in\mathbf{R}^3\,|\,\phi(x;p,p')=c\}.$$

This is a *spheroid* with focal points p and p'. It is a compact surface of class C^∞.

Since D is contained in the *exterior* of spheroid $E_c(p,p')$ with $c=\min_{x\in\partial D}\phi(x;p,p')$, Theorems 1.1 gives us the largest spheroid with focal points p and p' whose exterior contains D using dynamical bistatic data u_f on $B'\times\,]0,\,T[$. The appearance of the spheroid in the enclosure method is new and this is a decisive difference from the previous enclosure method.

Therefore we obtain the information that there exists a point belonging to ∂D on the spheroid $E_c(p,p')$ with $c=\min_{x\in\partial D}\phi(x;p,p')$ calculated by formula (1.6). Thus, the next problem is: identify all the points belonging to ∂D on the spheroid. In order to describe the problem precisely we introduce the following notion.

Definition 1.1. Let p and p' satisfy $[p,p']\cap\partial D=\emptyset$. Define

$$\Lambda_{\partial D}(p,p')=\{q\in\partial D\,|\,\phi(q;p,p')=\min_{x\in\partial D}\phi(x;p,p')\}.$$

We call this the *first reflector* between p and p'. The points in the first reflector are called the *first reflection points* between p and p'. Note that $\Lambda_{\partial D}(p,p')$ can be an infinite set.

One has the expression

$$\Lambda_{\partial D}(p,p')=\partial D\cap E_c(p,p')$$

with $c=\min_{x\in\partial D}\phi(x;p,p')$. Thus the problem becomes: identify all the first reflection points.

Let $\omega\in S^2$. We denote by $s(\omega;p,p',c)$ the length of the straight line segment connecting p' and the unique point on $E_c(p,p')\cap\{\,p'+s\omega\,\vert\,s>0\,\}$. We have

$$s(\omega;p,p',c)=\frac{c^2-\vert p-p'\vert^2}{2\{c-\omega\cdot(p-p')\}}.$$

Note that $\omega\cdot(p-p')<c$ since $c>\vert p-p'\vert$. It is easy to see that the map

$$S^2\ni\omega\mapsto p'+s(\omega;p,p',c)\omega\in\mathbf{R}^3$$

is one-to-one and the image coincides with $E_c(p,p')$.

Let $0<\eta'<\inf_{\omega\in S^2}s(\omega;p,p',c)$. $\overline{B}'$ is contained in the set of all x such that $\phi(x;p,p')<c$, that is, the domain enclosed by $E_c(p,p')$.

The following theorem says that all the first reflection points between p and p' together with the tangent planes can be extracted from a single set of the bistatic data. This exceeds the previous enclosure method and suggests that the information which contained in the bistatic data is quite rich.

Theorem 1.2. *Assume that $c=\min_{x\in\partial D}\phi(x;p,p')$ is known. Let $[\overline{B}\cup\overline{B}']\cap\overline{D}=\emptyset$. Fix $0<s<\eta'$. If T satisfies*

$$T>\sup_{\omega\in S^2}\min_{x\in\partial D}\phi(x;p,p'+s\omega)-(\eta+\eta'-s),\tag{1.8}$$

then, one can extract all $q\in\Lambda_{\partial D}(p,p')$ together with ν_q from u_f on $B'\times\,]0,\,T[$ with $f=\chi_B$.

Remark 1.1. Note that

$$\sup_{\omega\in S^2}\min_{x\in\partial D}\phi(x;p,p'+s\omega)-(\eta+\eta'-s)=\sup_{\omega\in S^2}\min_{x\in\partial D,\,y\in\partial B,\,y'\in\partial B_{\eta'-s}(p'+s\omega)}\phi(x;y,y').$$

The reason is the same as that of the validity of (1.7). Thus the constraint on T is reasonable.

Theorem 1.1 is a direct consequence of the following two estimates: there exist $\mu_j\in\mathbf{R}$, $C_j>0$ with $j=1,2$ and $\tau_0>0$ which are independent of τ such that, for all $\tau\ge\tau_0$,

$$e^{\tau\min_{x\in\partial D,\,y\in\partial B,\,y'\in\partial B'}\phi(x;y,y')}\int_{\mathbf{R}^3\setminus\overline{D}}(fv_g-w_fg)dx\le C_1\tau^{\mu_1}\tag{1.9}$$

and

$$C_2\tau^{\mu_2}\le e^{\tau\min_{x\in\partial D,\,y\in\partial B,\,y'\in\partial B'}\phi(x;y,y')}\int_{\mathbf{R}^3\setminus\overline{D}}(fv_g-w_fg)dx.\tag{1.10}$$

The proof of (1.9) proceeds along the same line as the back-scattering data case ($B = B'$) and is given in Section 2. The point that should be emphasized in the proof of Theorem 1.1 is (1.10) which is proved in Subsection 3.1.

When $B' = B$, using the same technique as done for the *sound-hard* obstacle case in [11], we can prove (1.10) without difficulty. The technique therein does not depend on the boundary condition, however, heavily depends on the condition $B = B'$. In this paper, we take another way. It is based on the combination of the *maximum principle* for the modified Helmholtz equation in the domain $\mathbf{R}^3 \setminus \overline{D}$ and a *reflection* across ∂D. It heavily depends on the speciality of the homogeneous Dirichlet boundary condition on ∂D. The idea goes back to the arguments done in the proofs of Theorem 3.6 and Lemma 3.7 in Lax-Phillips [18]. Note that, therein, a relationship between the *support function* and the so-called *scattering kernel* for a general sound-soft obstacle has been established. They used the arguments to obtain an estimate for the *analytic continuation* of the Fourier transform of the scattering kernel and then applied the Paley-Weiner theorem. We refer the reader to [19, 22] for several other results using the scattering kernel.

1.2 Leading term of the indicator function and its implication

(1.9) and (1.10) suggest that the following integral as $\tau \longrightarrow \infty$

$$e^{\tau \min_{x\in\partial D,\, y\in\partial B,\, y'\in\partial B'} \phi(x;y,y')} \int_{\mathbf{R}^3\setminus\overline{D}} (fv_g - w_f g)dx$$

may behave as some power of τ multiplied by a positive constant. The constant may contain some information about the geometry of the boundary of the obstacle at the points on ∂D that attain $\min_{x\in\partial D,\, y\in\partial B,\, y'\in\partial B'} \phi(x;y,y')$, i.e., the first reflection points between p and p'.

If $q \in \Lambda_{\partial D}(p,p')$, then $q \in E_c(p,p')$ with $c = \min_{x\in\partial D} \phi(x;p,p')$ and the two tangent planes at q of ∂D and $E_c(p,p')$ coincide. We denote by $S_q(\partial D)$ and $S_q(E_c(p,p'))$ the *shape operators* (or the *Weingarten maps*) at q with respect to ν_q. Those are symmetric linear operators on the common tangent space at q of ∂D and $E_c(p,p')$. It is easy to see that $S_q(E_c(p,p')) - S_q(\partial D) \geq 0$ as the quadratic form on the same tangent space at q (see (4.21)).

Let $Z_f \in H^1(\mathbf{R}^3 \setminus \overline{D})$ solve

$$\begin{aligned} (\triangle - \tau^2) Z_f &= F_f(x,\tau) \text{ in } \mathbf{R}^3 \setminus \overline{D}, \\ Z_f &= 0 \text{ on } \partial D. \end{aligned} \tag{1.11}$$

It follows from (1.2) for $h = f$ and (1.4) that w_f has the form

$$w_f = v_f + \epsilon_f^0 + e^{-\tau T} Z_f, \tag{1.12}$$

where ϵ_f^0 satisfies

$$\begin{aligned} (\triangle - \tau^2)\epsilon_f^0 &= 0 \text{ in } \mathbf{R}^3 \setminus \overline{D}, \\ \epsilon_f^0 &= -v_f \text{ on } \partial D. \end{aligned} \tag{1.13}$$

Note that: since $\operatorname{supp} f \cap \overline{D} = \emptyset$, v_f is smooth in a neighbourhood of $\overline{D}$ and thus, by elliptic regularity, we see that ϵ_f^0 is smooth for $x \in \mathbf{R}^3 \setminus D$. Moreover, note that $\epsilon_f^0(x) \longrightarrow 0$

as $|x| \longrightarrow \infty$ rapidly and uniformly with respect to $x/|x|$. This is a combination of the uniqueness of the weak solution of (1.13) and a potential theoretic construction of the solution, see, e.g., [3, 20] for the approach and [15] for an application to an inverse problem for the heat equation.

Given $x \in \mathbf{R}^3$ define $d_{\partial D}(x) = \inf_{y\in\partial D}|y-x|$. It is well known that there exists a positive constant δ_0 such that: given $x \in \overline{D}/x \in \mathbf{R}^3 \setminus D$ with $d_{\partial D}(x) < 2\delta_0$ there exists a unique $q = q(x)$ be the boundary point on ∂D such that $x = q \mp d_{\partial D}(x)\nu_q$ ([6]). One may assume that both $d_{\partial D}(x)$ and $q(x)$ is C^2 for $x \in \overline{D}$ with $d_{\partial D}(x) < 2\delta_0$; $x \in \mathbf{R}^3 \setminus D$ with $d_{\partial D}(x) < 2\delta_0$ (Lemma 1 of Appendix in [6]). Note that ν_q is the unit outer normal to ∂D at q. For x with $d_{\partial D}(x) < 2\delta_0$ define $x^r = 2q(x) - x$.

Before describing our third result, we introduce a restriction on a class of obstacles which is satisfied with all convex obstacles.

Definition 1.2. We say that D is *admissible*, if there exist positive constants C, $\delta'(\le 2\delta_0)$ and τ_0 such that, for all $y \in D$ with $d_{\partial D}(y) < \delta'$ and $\tau \ge \tau_0$

$$|\epsilon_f^0(y^r)| \le C\int_B e^{-\tau|y-x|}dx.$$

The following theorem gives an answer to the question raised above.

Theorem 1.3. *Let B and B' satisfy $[\overline{B}\cup\overline{B'}]\cap\overline{D} = \emptyset$. Let $f = \chi_B$ and $g = \chi_{B'}$. Let $c = \min_{x\in\partial D}\phi(x;p,p')$. Let T satisfy (1.5).*

Assume that D is admissible and ∂D is C^3. If $\Lambda_{\partial D}(p,p')$ is finite and for all $q \in \Lambda_{\partial D}(p,p')$

$$det\,(S_q(E_c(p,p')) - S_q(\partial D)) > 0, \tag{1.14}$$

then we have

$$\lim_{\tau\longrightarrow\infty}\tau^4 e^{\tau\min_{x\in\partial D,\,y\in\partial B,\,y'\in\partial B'}\phi(x;y,y')}\int_{\mathbf{R}^3\setminus\overline{D}}(fv_g - w_f g)dx$$
$$= \frac{\pi}{2}\sum_{q\in\Lambda_{\partial D}(p,p')}\left(\frac{diam\,B}{2|q-p|}\right)\cdot\left(\frac{diam\,B'}{2|q-p'|}\right)\cdot\frac{1}{\sqrt{det\,(S_q(E_c(p,p')) - S_q(\partial D))}}. \tag{1.15}$$

Some remarks are in order.

• The right-hand side of (1.15) is symmetric with respect to the replacement $p \to p'$ and $p' \to p$. This is a kind of *reciprocity*.

• The quantity $\det\,(S_q(E_c(p,p')) - S_q(\partial D))$ expresses some kind of information about the difference or deviation of the geometry between ∂D and $E_c(p,p')$ at $q \in \Lambda_{\partial D}(p,p')$.

The following proposition says that Theorem 1.3 can cover convex obstacles.

Proposition 1.1. *(i) If D is convex, then D is admissible and $\Lambda_{\partial D}(p,p')$ consists of a single point. (ii) Let $q \in \Lambda_{\partial D}(p,p')$ and assume that ∂D is contained in the half space $(x-q)\cdot\nu_q \le 0$, then (1.14) at q is satisfied.*

For the proof see Appendix. Thus as a corollary we obtain the following result.

Corollary 1.1. *Let B and B' satisfy $[\overline{B}\cup\overline{B'}]\cap\overline{D} = \emptyset$. Let $f = \chi_B$ and $g = \chi_{B'}$. Let $c = \min_{x\in\partial D}\phi(x;p,p')$. Let T satisfy (1.5). If D is convex and ∂D is C^3, then (1.15) whose right-hand side consists of a single term is valid.*

Note that, for the back-scattering case $B=B'$, using the completely same argument as done in [12] in a bounded domain, we obtain

$$\lim_{\tau\longrightarrow\infty}\tau^4 e^{2\tau\text{dist}\,(\partial D,B)}\int_B(v_f-w_f)dx$$
$$=\frac{\pi}{2}\left(\frac{\text{diam}\,B}{2d_{\partial D}(p)}\right)^2\sum_{x\in\partial D,\,\vert x-p\vert=d_{\partial D}(p)}\frac{1}{\sqrt{P_{\partial D}(1/d_{\partial D}(p);x)}},\tag{1.16}$$

where $d_{\partial D}(p)=\inf_{x\in\partial D}\vert x-p\vert$; $P_{\partial D}(\lambda;q)=(\lambda-k_1(q))\,(\lambda-k_2(q))$ and $k_1(q)$ and $k_2(q)$ denote the *principle curvatures* of ∂D at q with respect to ν_q (see Appendix in [6]). Note that the Gauss curvature $K_{\partial D}(q)$ and mean curvature $H_{\partial D}(q)$ at q with respect to ν_q are given by $k_1(q)k_2(q)$ and $(k_1(q)+k_2(q))/2$, respectively.

The assumptions therein are

• ∂D is C^3;

• $T>2\text{dist}\,(D,B)$;

• the set of all points $x\in\partial D$ with $\vert x-p\vert=d_{\partial D}(p)$ is *finite* and each point q in the set satisfies $P_{\partial D}(1/d_{\partial D}(p);q)>0$.

It is not assumed that D is admissible in (1.16) unlike (1.15).

The quantity $P_{\partial D}(1/d_{\partial D}(p);q)$ at $q\in\partial D$ with $\vert q-p\vert=d_{\partial D}(p)$ denotes a 'deflection' of the surface ∂D at q from the sphere $\vert x-p\vert=d_{\partial D}(p)$ since we know from, e.g., Proposition 4.2 in this paper that $P_{\partial D}(q;1/d_D(p))=\det\,(S_q(\partial B_{d_D(p)}(p))-S_q(\partial D))$, where $B_{d_D(p)}(p)=\{x\in\mathbf{R}^3\,\vert\,\vert x-p\vert<d_D(p)\}$. Thus formula (1.15) of Theorem 1.3 can be considered as an extension of (1.16) to the bistatic data case. See also Remark 5.2 for a comparison.

After having Theorem 1.3, everyone wishes to extract the geometry of ∂D at all the first reflection points. The complete answer for general obstacle is not known, however, under the admissibility of D, one can obtain the following result.

Theorem 1.4. *Let $q\in\Lambda_{\partial D}(p,p')$ be known. Let T satisfy (1.5). Let B and B' satisfy $[\overline{B}\cup\overline{B}']\cap\overline{D}=\emptyset$. If D is admissible and ∂D is C^3, then one can extract*

$$H_{\partial D}(q)-\frac{S_q(\partial D)(\boldsymbol{A}_q(p)\times\boldsymbol{A}_q(p'))\cdot(\boldsymbol{A}_q(p)\times\boldsymbol{A}_q(p'))}{2(1+\boldsymbol{A}_q(p)\cdot\boldsymbol{A}_q(p'))}\tag{1.17}$$

and

$$K_{\partial D}(q)$$

from u_f on $B'\times\,]0,\,T[$ with $f=\chi_B$, where

$$\boldsymbol{A}_q(x)=\frac{q-x}{\vert q-x\vert}.$$

Note that $\boldsymbol{A}_q(p)\times\boldsymbol{A}_q(p')$ belongs to the tangent space of ∂D at q. For this see Lemma 4.3 in Section 4.

This theorem may suggest the following.

• If one wishes to know the mean curvature at a first reflection point precisely, one should make the transmitter and the receiver approach as much as possible. It is because

$\boldsymbol{A}_q(p)\times\boldsymbol{A}_q(p')$ will disappear approximately at this time and thus the correction term in (1.17) can be ignored.

• On the other hand, the Gauss curvature at the first reflection point can be extracted regardless of the position of a transmitter and a receiver at any time except for the condition $[\overline{B}\cup\overline{B}']\cap\overline{D}=\emptyset$.

As a corollary of Theorems 1.1, 1.2 and 1.4 we have the following result.

Corollary 1.2. *Assume that D is an open ball. Let T satisfy (1.8). Let B and B' satisfy $[\overline{B}\cup\overline{B}']\cap\overline{D}=\emptyset$. Then, one can extract D itself from u_f on $B'\times\,]0,\,T[$ with $f=\chi_B$.*

The steps to reconstruct an unknown open ball D are as follows.

Step 1. Determine $c=\min_{x\in\partial D}\phi(x;p,p')$ via Theorem 1.1.

Step 2. Determine the unique point q in $\Lambda_{\partial D}(p,p')$ together with ν_q via Theorem 1.2.

Step 3. Determine $K_{\partial D}(q)$ via Theorem 1.4.

Then the radius and center of D are given by $1/\sqrt{K_{\partial D}(q)}$ and $q-(1/\sqrt{K_{\partial D}(q)})\nu_q$, respectively.

The reconstruction problem of a spherical obstacle also has been considered in the frequency domain. For example, see [1] which employs a spherical wave as an incident wave and uses a low frequency limit for the reconstruction.

The four steps described above give a *constructive proof* of a uniqueness theorem in an inverse obstacle problem in the sense that it does not make use of the *uniqueness of the continuation* of the solution of the governing equation of the wave. The following uniqueness result employs the bistatic data over a *finite time interval* and itself seems to be new.

Theorem 1.5. *Let D_1 and D_2 be open balls. Let u_f^j be the solution of (1.1) with $f=\chi_B$ and $D=D_j$. Let T satisfy (1.8). Let B and B' satisfy $[\overline{B}\cup\overline{B}']\cap\overline{D}_j=\emptyset$ for $j=1,2$. If $u_f^1=u_f^2$ on $B'\times\,]0,\,T[$, then $D_1=D_2$.*

We refer the readers to [16, 17, 23, 24] for various uniqueness theorems for inverse obstacle problems for hyperbolic equations over a finite time interval.

Another corollary from Theorem 1.4 is concerned with the determination of the directions of principle curvatures at a point on ∂D.

Assume that D is convex and ∂D is C^3. From Proposition 1.1 we know that $\Lambda_{\partial D}(p,p')$ consists of a single point. We denote the point by $q(p,p')$. We denote by $p(\theta)$ and $p'(\theta)$ the points rotated around the line directed ν_q at $q=q(p,p')$ counterclockwise with rotation angle $\theta\in[0,\,2\pi[$ of p and p'. Thus $p(0)=p$ an $p'(0)=p'$.

Then for all $\theta\in\,[0,\,2\pi[$ we know that $\Lambda_{\partial D}(p(\theta),p'(\theta))$, $\boldsymbol{A}_q(p(\theta))\cdot\boldsymbol{A}_q(p'(\theta))$, $\vert\boldsymbol{A}_q(p(\theta))\times\boldsymbol{A}_q(p'(\theta))\vert$, $\phi(q(\theta);p(\theta),p'(\theta))$ and ν_q at $q=q(p(\theta),p'(\theta))$ are invariant with respect to θ.

Let $B(\theta)$ denote the open ball centered at $p(\theta)$ with radius η and $B'(\theta)$ the open ball centered at $p'(\theta)$ with radius η. We have $[\overline{B}(\theta)\cup\overline{B}'(\theta)]\cap\overline{D}=\emptyset$ provided $[\overline{B}\cup\overline{B}']\cap\overline{D}=\emptyset$ and D is convex.

Then from (1.17) in Theorem 1.4 applied to $f=f(\theta)$ and $B=B(\theta)$ and $B'=B'(\theta)$ we obtain the function of θ:

$$\theta\longmapsto\tilde{H}_{\partial D}(q;p(\theta),p'(\theta))\equiv H_{\partial D}(q)-\frac{1}{2}\sqrt{\frac{1-\boldsymbol{A}_q(p)\cdot\boldsymbol{A}_q(p')}{1+\boldsymbol{A}_q(p)\cdot\boldsymbol{A}_q(p'))}}S_q(\partial D)(\boldsymbol{V}(\theta))\cdot\boldsymbol{V}(\theta)$$

where $\boldsymbol{V}(\theta)$ denotes the unit vector directed to $\boldsymbol{A}_q(p(\theta))\times\boldsymbol{A}_q(p'(\theta))$.

Now assume that $\boldsymbol{A}_q(p) \times \boldsymbol{A}_q(p') \neq 0$. Then, $\boldsymbol{V}(\theta)$ attains all the tangent vector at q of ∂D and thus from the behaviour of $\tilde{H}_{\partial D}(q; p(\theta), p'(\theta))$ as a function of θ one can determine all the directions of principle curvatures say, $\boldsymbol{V}(\theta_1)$ and $\boldsymbol{V}(\theta_2)$ with some θ_1 and θ_2. Then we have

$$\frac{\tilde{H}_{\partial D}(q; p(\theta_1), p'(\theta_1)) + \tilde{H}_{\partial D}(q; p(\theta_2), p'(\theta_2))}{2} = \left\{ 1 - \frac{1}{2}\sqrt{\frac{1 - \boldsymbol{A}_q(p) \cdot \boldsymbol{A}_q(p')}{1 + \boldsymbol{A}_q(p) \cdot \boldsymbol{A}_q(p'))}} \right\} H_{\partial D}(q).$$

Thus we obtain $H_{\partial D}(q)$.

Summing up, we have obtained the following result.

Corollary 1.3. *Let B and B' satisfy $[\overline{B} \cup \overline{B}'] \cap \overline{D} = \emptyset$. Let T satisfy (1.5). Assume that D is convex and ∂D is C^3; $q = q(p, p')$ is known; $\boldsymbol{A}_q(p) \times \boldsymbol{A}_q(p') \neq 0$. Then, one can extract all the directions of principle curvatures, mean and Gauss curvatures, in other words, the shape operator at q of ∂D from $u_{f(\theta)}$ over $B'(\theta) \times]0,\, T[$ for all $\theta \in [0,\, 2\pi[$, where $f(\theta)$ denotes the characteristic function of $B(\theta)$.*

A brief outline of this paper is as follows. Theorems 1.1 is proved in Sections 2 and 3. As described above, the key point of the proof is to derive (1.9) and (1.10) and those are proved in Sections 2 and 3, respectively.

Theorem 1.2 is proved in Subsection 5.1. The proof contains an explicit characterization of the first reflector in terms of the bistatic data. See Remark 5.1 for the resulted procedure to determine all the first reflection points.

Theorem 1.3 is proved in Section 4. The key point in the proof of (1.15) as well as (1.16) is to identify the term which contains the leading term of the indicator function. See (4.1) for the term. We found that the one of two reflection arguments developed in [18] works for the purpose. It is based on the reflection across ∂D and a pointwise estimate of ϵ_f^0 near ∂D, that is the use of the admissibility of D. The argument is presented in the proof of Lemma 4.2 in Section 4. Note that another reflection argument used in the proof of (1.16) is free from the admissibility assumption, however, can not be applied to the case when $f \not\equiv g$.

Theorem 1.4 is proved in Subsection 5.2. The proof is based on an asymptotic formula which is a consequence of Theorem 1.3 and an explicit formula of the determinant of the difference of two shape operators at $q \in \Lambda_{\partial D}(p, p')$ as derived in Subsection 7.3 of Appendix.

In the final section we give a conclusion of this paper and comments on further problems.

2 An upper bound of the indicator function

Define

$$J(\tau; f, g) = \int_D (\nabla v_f \cdot \nabla v_g + \tau^2 v_f v_g) dx. \tag{2.1}$$

We have the following expression of the indicator function.

Proposition 2.1. *It follows that*

$$
\begin{aligned}
\int_{\mathbf{R}^3\setminus\overline{D}}(fv_g-w_fg)dx=J(\tau;f,g)+\int_{\mathbf{R}^3\setminus\overline{D}}(\nabla\epsilon_f^0\cdot\nabla\epsilon_g^0+\tau^2\epsilon_f^0\epsilon_g^0)dx\\
-e^{-\tau T}\int_{\mathbf{R}^3\setminus\overline{D}}(\nabla Z_f\cdot\nabla v_g+\tau^2 Z_fv_g)dx.
\end{aligned}
\tag{2.2}
$$

Proof. From (1.2) and (1.4) we have

$$
\int_{\mathbf{R}^3\setminus\overline{D}}(fv_g-w_fg)dx=\int_{\partial D}\frac{\partial w_f}{\partial\nu}v_gdS+e^{-\tau T}\int_{\mathbf{R}^3\setminus\overline{D}}F_f(x,\tau)v_gdx.
\tag{2.3}
$$

Rewrite

$$
\int_{\partial D}\frac{\partial w_f}{\partial\nu}v_gdS=\int_{\partial D}\frac{\partial v_f}{\partial\nu}v_gdS+\int_{\partial D}\frac{\partial\epsilon_f^0}{\partial\nu}v_gdS+e^{-\tau T}\int_{\partial D}\frac{\partial Z_f}{\partial\nu}v_gdS.
\tag{2.4}
$$

Integration by parts yields

$$
\int_{\partial D}\frac{\partial v_f}{\partial\nu}v_gdS=J(\tau;f,g).
\tag{2.5}
$$

On the other hand, (1.13) yields

$$
\int_{\partial D}\frac{\partial\epsilon_f^0}{\partial\nu}v_gdS=-\int_{\partial D}\frac{\partial\epsilon_f^0}{\partial\nu}\epsilon_g^0dS=\int_{\mathbf{R}^3\setminus\overline{D}}(\nabla\epsilon_f^0\cdot\nabla\epsilon_g^0+\tau^2\epsilon_f^0\epsilon_g^0)dx.
\tag{2.6}
$$

Furthermore it follows from (1.2) and (1.11) that

$$
-\int_{\partial D}\frac{\partial Z_f}{\partial\nu}v_gdS=\int_{\mathbf{R}^3\setminus\overline{D}}(\nabla Z_f\cdot\nabla v_g+\tau^2Z_fv_g+F_fv_g)dx.
$$

Now from this together with (2.3)-(2.6) we obtain (2.2).
□

Lemma 2.1. *As* $\tau\longrightarrow\infty$

$$
\left|\int_{\mathbf{R}^3\setminus\overline{D}}(\nabla Z_f\cdot\nabla v_g+\tau^2Z_fv_g)dx\right|=O(\tau^{-1}).
\tag{2.7}
$$

Proof. From (1.2), we obtain

$$
\int_{\mathbf{R}^3}\left(|\nabla v_g|^2+\tau^2\left|v_f-\frac{g}{2\tau^2}\right|^2\right)dx=\frac{1}{4\tau^2}\int_B|g|^2dx.
$$

Since

$$
\left|v_g-\frac{g}{2\tau^2}\right|^2\ge\frac{1}{2}|v_g|^2-\frac{|g|^2}{4\tau^4},
$$

from this we obtain

$$
\frac{1}{2}\int_{\mathbf{R}^3}(|\nabla v_g|^2+\tau^2|v_g|^2)dx\le\frac{1}{2\tau^2}\int_B|g|^2dx
$$

and thus

$$\int_{\mathbf{R}^3}(|\nabla v_g|^2+\tau^2|v_g|^2)dx \le \frac{1}{\tau^2}\int_B |g|^2 dx. \tag{2.8}$$

Similarly it follows from (1.11) that

$$\int_{\mathbf{R}^3\setminus\overline{D}}(|\nabla Z_f|^2+\tau^2|Z_f|^2)dx \le \frac{1}{\tau^2}\int_{\mathbf{R}^3\setminus\overline{D}} |F_f|^2 dx. \tag{2.9}$$

A combination of (2.8) and (2.9) and the estimate $\|F_f\|_{L^2(\mathbf{R}^3\setminus\overline{D})}=O(\tau)$ yields (2.7).
□

Thus a combination of (2.2) and (2.7) gives

$$\int_{\mathbf{R}^3\setminus\overline{D}}(fv_g-w_f g)dx = J(\tau;f,g)+\int_{\mathbf{R}^3\setminus\overline{D}}(\nabla\epsilon_f^0\cdot\nabla\epsilon_g^0+\tau^2\epsilon_f^0\epsilon_g^0)dx+O(\tau^{-1}e^{-\tau T}). \tag{2.10}$$

For the second term in this right-hand side we have the following estimate.

Lemma 2.2. *As* $\tau\longrightarrow\infty$

$$\int_{\mathbf{R}^3\setminus\overline{D}}(|\nabla\epsilon_f^0|^2+\tau^2|\epsilon_f^0|^2)dx = O(\tau^2 J(\tau;f,f)). \tag{2.11}$$

Proof. It is an application of the trace theorem twice and integration by parts. More precisely, choose $\tilde{v}\in H^1(\mathbf{R}^3\setminus\overline{D})$ in such a way that

$$\|\tilde{v}\|_{H^1(\mathbf{R}^3\setminus\overline{D})} \le C\|v_f|_{\partial D}\|_{H^{1/2}(\partial D)}, \tag{2.12}$$

where $C>0$ and is independent of v_f. Integration by parts (or the weak formulation of (1.13)) yields

$$\int_{\partial D}\frac{\partial\epsilon_f^0}{\partial\nu}v_f dS = -\int_{\mathbf{R}^3\setminus\overline{D}}(\nabla\epsilon_f^0\cdot\nabla\tilde{v}+\tau^2\epsilon_f^0\tilde{v})dx. \tag{2.13}$$

A combination (2.12) and (2.13) gives

$$\left|\int_{\partial D}\frac{\partial\epsilon_f^0}{\partial\nu}v_f dS\right| \le C(\|\nabla\epsilon_f^0\|_{L^2(\mathbf{R}^3\setminus\overline{D})}+\tau^2\|\epsilon_f^0\|_{L^2(\mathbf{R}^3\setminus\overline{D})})\|v_f|_{\partial D}\|_{H^{1/2}(\partial D)}. \tag{2.14}$$

Since

$$\max\left(\|\nabla\epsilon_f^0\|_{L^2(\mathbf{R}^3\setminus\overline{D})},\tau\|\epsilon_f^0\|_{L^2(\mathbf{R}^3\setminus\overline{D})}\right) \le \left(\int_{\mathbf{R}^3\setminus\overline{D}}(|\nabla\epsilon_f^0|^2+\tau^2|\epsilon_f^0|^2)dx\right)^{1/2},$$

it follows from (2.14) that

$$\left|\int_{\partial D}\frac{\partial\epsilon_f^0}{\partial\nu}v_f dS\right| \le C(1+\tau)\left(\int_{\mathbf{R}^3\setminus\overline{D}}(|\nabla\epsilon_f^0|^2+\tau^2|\epsilon_f^0|^2)dx\right)^{1/2}\|v_f|_{\partial D}\|_{H^{1/2}(\partial D)}.$$

From this together with (2.6) for $f=g$, we obtain

$$\int_{\mathbf{R}^3\setminus\overline{D}}(|\nabla\epsilon_f^0|^2+\tau^2|\epsilon_f^0|^2)dx \le C^2(1+\tau)^2\|v_f|_{\partial D}\|_{H^{1/2}(\partial D)}^2. \tag{2.15}$$

By the trace theorem, we have

$$\|v_f|_{\partial D}\|^2_{H^{1/2}(\partial D)} \le C'(\|\nabla v_f\|^2_{L^2(D)} + \|v_f\|^2_{L^2(D)}),$$

where $C' > 0$ is independent of v_f. Now from this together with (2.15) and the trivial estimates

$$\max\,(\|\nabla v_f\|^2_{L^2(D)}, \tau^2\|v_f\|^2_{L^2(D)}) \le J(\tau; f, f),$$

yields (2.11).
□

Therefore (1.9) with $\mu_1 = 2$ follows from (2.10) and (2.11) together with the following estimate.

Lemma 2.3. *We have, as*$\tau \longrightarrow \infty$*,*

$$J(\tau; f, g) = O(\tau e^{-\tau \min_{x\in\partial D,\, y\in\partial B,\, y'\in\partial B'} \phi(x;y,y')}). \tag{2.16}$$

Proof. It follows from (1.3) and (2.5) that

$$J(\tau; f, g) = \left(\frac{1}{4\pi}\right)^2 \int_{\partial D} dS_x \int_{B\times B'} k_\tau(x, y, y')dydy', \tag{2.17}$$

where

$$k_\tau(x, y, y') = \left(\frac{1}{|x-y|} + \tau\right)\frac{(y-x)\cdot\nu_x}{|x-y|^2|x-y'|}e^{-\tau(|x-y|+|x-y'|)},\ (x, y, y') \in \partial D \times B \times B'. \tag{2.18}$$

Since we have

$$\inf_{x\in\partial D,\, y\in B,\, y'\in B'} \phi(x; y, y') = \min_{x\in\partial D,\, y\in\partial B,\, y'\in\partial B'} \phi(x; y, y')$$

and $\overline{B} \cap \overline{D} = \overline{B}' \cap \overline{D} = \emptyset$, from (2.17) and (2.18) we obtain (2.16).
□

3 A lower bound of the indicator function

3.1 A reduction to a convex obstacle and the proof of (1.10).

Rewriting the second term in the right-hand side of (2.10) with (2.6), one has

$$\int_{\mathbf{R}^3\setminus\overline{D}} (fv_g - w_f g)dx = J(\tau; f, g) + \int_{\partial D} \frac{\partial\epsilon^0_f}{\partial\nu} v_g dS + O(\tau^{-1}e^{-\tau T}). \tag{3.1}$$

In general, we do not know the signature of the second term of the right-hand side of (3.1), however, we know that the function under integral is nonnegative at a special point on ∂D by virtue of the following lemma which is an application of the maximum principle for differential operator $\triangle - \tau^2$ and a reflection argument in [18]. It corresponds to Lemma 3.7 in [18] in which v_f in (1.13) is replaced with $-e^{-\tau x\cdot\omega}$ for a $\omega \in S^2$.

Lemma 3.1. *Let $q \in \partial D$ be a point of support of D, i.e., D is contained in the half-space $x \cdot n_q < q \cdot n_q$. We have*

$$\frac{\partial \epsilon_f^0}{\partial \nu}(q) \geq 0. \tag{3.2}$$

Proof. First, we prove that, for all $x \in \mathbf{R}^3 \setminus \overline{D}$,

$$\epsilon_f^0(x) \geq -v_f(x). \tag{3.3}$$

Define $w_f^0 = \epsilon_f^0 + v_f \in H^1(\mathbf{R}^3 \setminus \overline{D})$. It holds that

$$(\triangle - \tau^2) w_f^0 = -f \text{ in } \mathbf{R}^3 \setminus \overline{D},$$

$$w_f^0 = 0 \text{ on } \partial D.$$

Since $(\triangle - \tau^2) w_f^0 \leq 0$ in $\mathbf{R}^3 \setminus \overline{D}$, from the weak maximum principle for operators of divergence form (Theorem 8.1 in [6]), we obtain

$$\inf_{(\mathbf{R}^3 \setminus \overline{D}) \cap B_R} w_f^0 \geq \min\,(\inf_{S_R} w_f^0, 0). \tag{3.4}$$

where B_R denotes an arbitrary open ball centered at the origin such that $\overline{D} \cup \text{supp}\, f \subset B_R$ and $S_R = \partial B_R$. Since ϵ_f^0 decays as $|x| \longrightarrow \infty$ uniformly with respect to $x/|x|$, we have $\inf_{S_R} w_f^0 \longrightarrow 0$ as $R \longrightarrow \infty$, and thus, from (3.4) we obtain $w_f^0(x) \geq 0$ for all $x \in \mathbf{R}^3 \setminus \overline{D}$. This completes the proof of (3.3).

The equality in (3.3) holds for $x = q$. This implies the following inequality for the normal derivatives:

$$\frac{\partial \epsilon_f^0}{\partial \nu}(q) \geq -\frac{\partial v_f}{\partial \nu}(q). \tag{3.5}$$

Second, we prove that, for all points that satisfy $x \cdot n_q \geq q \cdot n_q$,

$$\epsilon_f^0(x) \geq -v_f(x'), \tag{3.6}$$

where x' is the image of x under reflection across the plane $x \cdot n_q = q \cdot n_q$. Since $x = x'$ on the plane $x \cdot n_q = q \cdot n_q$, (3.3) shows that (3.6) is satisfied there. Note that also $v_f'(x) \equiv v_f(x')$ satisfies $(\triangle - \tau^2) v_f' = -f(x') \leq 0$. Applying the weak maximum principle to $(w_f^0)' \equiv \epsilon_f^0 + v_f'$ in the half-space $x \cdot n_q > q \cdot n_q$, one obtains as before (3.6) holds throughout the half-space. This completes the proof of (3.6).

Since the equality in (3.6) holds for $x = q$, it follows as before that

$$\frac{\partial \epsilon_f^0}{\partial \nu}(q) \geq -\frac{\partial}{\partial \nu}\left\{v_f(x')\right\}|_{x=q} = \frac{\partial v_f}{\partial \nu}(q). \tag{3.7}$$

Now a combination of (3.5) and (3.7) yields (3.2).
□

The following lemma is an easy consequence of the C^2-regularity of ∂D and thus the proof is omitted.

Lemma 3.2. *Let $q \in \Lambda_{\partial D}(p, p')$. Then, there exists an open ball $\tilde{D}$ contained in D such that $q \in \Lambda_{\partial \tilde{D}}(p, p')$ and thus $\min_{x \in \partial \tilde{D}} \phi(x; p, p') = \min_{x \in \partial D} \phi(x; p, p')$.*

Let $\tilde{u} = \tilde{u}_f$ denote the weak solution of the following initial boundary value problem:

$$\begin{aligned}
&\partial_t^2 \tilde{u} - \triangle \tilde{u} = 0 \text{ in} \, (\mathbf{R}^3 \setminus \overline{\tilde{D}}) \times \,]0, \, T[,\\
&\tilde{u}(x, 0) = 0 \text{ in} \, \mathbf{R}^3 \setminus \overline{\tilde{D}},\\
&\partial_t \tilde{u}(x, 0) = f(x) \text{ in} \, \mathbf{R}^3 \setminus \overline{\tilde{D}},\\
&\tilde{u} = 0 \text{ on} \, \partial\tilde{D} \times \,]0, \, T[.
\end{aligned} \tag{3.8}$$

Define

$$\tilde{w}_f(x, \tau) = \int_0^T e^{-\tau T} \tilde{u}(x, t) dt, \ \ x \in \mathbf{R}^3 \setminus \overline{\tilde{D}}, \, \tau > 0.$$

Lemma 3.3. *We have*

$$\int_{\mathbf{R}^3 \setminus \overline{D}} (f v_g - w_f g) dx \geq \int_{\mathbf{R}^3 \setminus \overline{\tilde{D}}} (f v_g - \tilde{w}_f g) dx + O(\tau^{-1} e^{-\tau T}). \tag{3.9}$$

Proof. Let $\tilde{Z}_f \in H^1(\mathbf{R}^3 \setminus \overline{\tilde{D}})$ solve

$$\begin{aligned}
&(\triangle - \tau^2) \tilde{Z}_f = \tilde{F}_f(x, \tau) \text{ in} \, \mathbf{R}^3 \setminus \overline{\tilde{D}},\\
&\tilde{Z}_f = 0 \text{ on} \, \partial\tilde{D},
\end{aligned}$$

where

$$\tilde{F}_f(x, \tau) = \partial_t \tilde{u}_f(x, T) + \tau \tilde{u}_f(x, T), \ \ x \in \mathbf{R}^3 \setminus \overline{\tilde{D}}.$$

Similar to Z_f which is the solution of (2.2), we have $\|\tilde{Z}_f\|_{L^2(\mathbf{R}^3 \setminus \overline{\tilde{D}})} = O(\tau^{-1})$. And similar to (1.12) for w_f, $\tilde{w}_f$ has the form

$$\tilde{w}_f = v_f + \tilde{\epsilon}_f^0 + e^{-\tau T} \tilde{Z}_f,$$

where $\tilde{\epsilon}_f^0$ satisfies

$$\begin{aligned}
&(\triangle - \tau^2) \tilde{\epsilon}_f^0 = 0 \text{ in} \, \mathbf{R}^3 \setminus \overline{\tilde{D}},\\
&\tilde{\epsilon}_f^0 = -v_f \text{ on} \, \partial\tilde{D}.
\end{aligned}$$

Thus, we have

$$\tilde{w}_f - w_f = (\tilde{\epsilon}_f^0 - \epsilon_f^0) + e^{-\tau T} (\tilde{Z}_f - Z_f) \text{ in} \, \mathbf{R}^3 \setminus \overline{D}.$$

Since $\tilde{v_f} \geq 0$ on $\partial\tilde{D}$ and $\tilde{\epsilon}_f^0(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$, by the maximum principle for the modified Helmholtz equation in $\mathbf{R}^3 \setminus \overline{\tilde{D}}$, we have $-\tilde{\epsilon}_f^0 \leq v_f$ in $\mathbf{R}^3 \setminus \tilde{D}$ and thus $-\tilde{\epsilon}_f^0 \leq -\epsilon_f^0$ on ∂D. Again by the maximum principle for the modified Helmholtz equation in $\mathbf{R}^3 \setminus \overline{D}$, we obtain $-\tilde{\epsilon}_f^0 \leq -\epsilon_f^0$ in $\mathbf{R}^3 \setminus D$. Therefore we obtain

$$\tilde{w}_f - w_f \geq e^{-\tau T} (\tilde{Z}_f - Z_f) \text{ in} \, \mathbf{R}^3 \setminus \overline{D}. \tag{3.10}$$

Since both supp g and supp f are contained in $\mathbf{R}^3\setminus\overline{D}$ and thus combining this with (3.10), we obtain

$$\int_{\mathbf{R}^3\setminus\overline{D}}(fv_g-w_fg)dx-\int_{\mathbf{R}^3\setminus\overline{\tilde{D}}}(fv_g-\tilde{w}_fg)dx\ge e^{-\tau T}\int_{\mathbf{R}^3\setminus\overline{D}}(\tilde{Z}_f-Z_f)gdx.$$

From the L^2-bounds for Z_f and $\tilde{Z}_f$, we see that this right-hand side has the bound $O(\tau^{-1}e^{-\tau T})$.
$\square$

Since $\tilde{D}$ is convex, every point $q\in\partial\tilde{D}$ is a point of support of $\tilde{D}$ and thus, from (3.1), (3.2) and (3.9) we obtain

$$\int_{\mathbf{R}^3\setminus\overline{D}}(fv_g-w_fg)dx\ge\tilde{J}(\tau;f,g)+O(\tau^{-1}e^{-\tau T}),\tag{3.11}$$

where

$$\tilde{J}(\tau;f,g)=\int_{\tilde{D}}(\nabla v_f\cdot\nabla v_g+\tau^2v_fv_g)dx.$$

Now everything is reduced to give a lower estimate for $\tilde{J}(\tau;f,g)$ as $\tau\longrightarrow\infty$. For this and the future use of it in the sound-hard obstacle case we give the estimate for $J(\tau;f,g)$ for general D.

In the following lemma we do not assume that D is convex.

Lemma 3.4. *There exist positive constants C, μ and τ_0 such that, for all $\tau\ge\tau_0$,*

$$\tau^{2+\mu}e^{\tau\min_{x\in\partial D}\phi(x;p,p')}e^{-\tau(\eta+\eta')}J(\tau;f,g)\ge C.\tag{3.12}$$

We give the proof of this lemma in the next subsection.

It follows from (3.11) and (3.12) for $D=\tilde{D}$ that there exist positive constants C' and $\tau_0'>0$ such that, for all $\tau\ge\tau_0'$,

$$\tau^{2+\mu}e^{\tau\min_{x\in\partial D,\,y\in\partial B,y'\in\partial B'}\phi(x;y,y')}\int_{\mathbf{R}^3\setminus\overline{D}}(fv_g-w_fg)dx\ge C'$$

provided T satisfies (1.5). This completes the proof of (1.10).

3.2 Proof of Lemma 3.4.

In this subsection we never assume that D is convex. Let $A(x,\tau)$ be an arbitrary positive function of $x\in\partial D$ with parameter $\tau>0$. For another function $B(x,\tau)$, in the following, $B(x,\tau)=O(A(x,\tau))$ as $\tau\longrightarrow\infty$ and uniformly with respect to $x\in\partial D$ means that there exist positive constants τ_0 and C independent of $x\in\partial D$ such that, for all $\tau\ge\tau_0$ and $x\in\partial D$ we have $|B(x,\tau)|\le CA(x,\tau)$.

The proof of Lemma 3.4 starts with having the following expression.

Lemma 3.5. *There exists a positive constant C such that, as $\tau\longrightarrow\infty$*

$$\begin{aligned}&J(\tau;f,g)\\&=\frac{1}{4\tau^3}\left(\eta-\frac{1}{\tau}\right)\left(\eta'-\frac{1}{\tau}\right)\int_{\partial D}\frac{(p-x)\cdot\nu_x}{|x-p|^2|x-p'|}\left(1+\frac{1}{\tau|x-p|}\right)e^{-\tau(d_B(x)+d_{B'}(x))}dS_x\\&\quad+O(\tau^{-1}e^{-\tau\inf_{x\in\partial D}(d_B(x)+d_{B'}(x))}(e^{-C\tau\min_{x\in\partial D}d_B(x)}+e^{-C\tau\min_{x\in\partial D}d_{B'}(x)})).\end{aligned}\tag{3.13}$$

Proof. By [12], we have, as $\tau \longrightarrow \infty$ and uniformly with respect to $x \in \partial D$,

$$v_g(x) = \frac{1}{2\tau}\left((J_1^+)_g(x,\tau) + O(e^{-\tau d_{B'}(x)(1+C)})\right)$$

and

$$\nabla v_f(x) = \frac{1}{2}\left((J_2^+)_f(x,\tau)\frac{p-x}{|x-p|} + O(e^{-\tau d_B(x)(1+C)})\right),$$

where C is a positive constant,

$$(J_2^+)_f(x,\tau) = \frac{e^{-\tau d_B(x)}}{\tau|x-p|}\left(\eta - \frac{1}{\tau}\right)\left(1 + \frac{1}{\tau|x-p|}\right)$$

$$+\frac{e^{-\tau d_B(x)\sqrt{1+(2\eta/d_B(x))}}}{\tau^2|x-p|^2}\left(d_B(x)\sqrt{1+\frac{2\eta}{d_B(x)}} + \frac{1}{\tau}\right)$$

and

$$(J_1^+)_g(x,\tau) = \frac{e^{-\tau d_{B'}(x)}}{\tau|x-p'|}\left(\eta' - \frac{1}{\tau}\right) + \frac{e^{-\tau d_{B'}(x)\sqrt{1+(2\eta'/d_{B'}(x))}}}{\tau^2|x-p'|}.$$

From these and (2.5) we obtain

$$J(\tau; f, g) = \frac{1}{4\tau}\int_{\partial D}(J_2^+)_f(x,\tau)\frac{(p-x)\cdot\nu_x}{|x-p|}(J_1^+)_g(x,\tau)dS_x$$

$$+O(\tau^{-1}e^{-\tau\inf_{x\in\partial D}(d_B(x)+d_{B'}(x))}(e^{-C\tau\min_{x\in\partial D}d_B(x)} + e^{-C\tau\min_{x\in\partial D}d_{B'}(x)})).$$

This yields (3.13) since we have as $\tau \longrightarrow \infty$ and uniformly with respect to $x \in \partial D$,

$$(J_1^+)_g(x,\tau)(J_2^+)_f(x,\tau) = \frac{e^{-\tau(d_B(x)+d_{B'}(x))}}{\tau^2|x-p||x-p'|}\left(\eta - \frac{1}{\tau}\right)\left(\eta' - \frac{1}{\tau}\right)\left(1 + \frac{1}{\tau|x-p|}\right)$$

$$+O(\tau^{-3}e^{-\tau(d_B(x)+d_{B'}(x))}(e^{-\tau d_{B'}(x)C} + e^{-\tau d_B(x)C})) + O(\tau^{-4}e^{-\tau(d_B(x)+d_{B'}(x))(1+C)}).$$

□

Now we give a lower estimate of $J(\tau; f, g)$ as $\tau \longrightarrow \infty$ by using (3.13).

Define

$$I_m(\tau) = \int_{\partial D}\frac{(p-x)\cdot\nu_x}{|x-p|^m|x-p'|}e^{-\tau\phi(x;p,p')}dS_x,$$

where $m = 2, 3$.

Since $d_B(x) = |x-p| - \eta$ for $x \in \mathbf{R}^3 \setminus B$ and $d_{B'}(x) = |x-p'| - \eta'$ for $x \in \mathbf{R}^3 \setminus B'$, it follows from (3.13) that

$$e^{-\tau(\eta+\eta')}J(\tau; f, g) = \frac{1}{4\tau^3}\left(\eta - \frac{1}{\tau}\right)\left(\eta' - \frac{1}{\tau}\right)\left(I_2(\tau) + \frac{1}{\tau}I_3(\tau)\right)$$
$$+O(\tau^{-1}e^{-\tau\min_{x\in\partial D}\phi(x;p,p')}(e^{-C\tau\min_{x\in\partial D}d_B(x)} + e^{-C\tau\min_{x\in\partial D}d_{B'}(x)})). \tag{3.14}$$

Since

$$\nabla\cdot\left\{\frac{(p-x)}{|x-p|^m|x-p'|}\right\} = \frac{(m-3)}{|x-p|^m|x-p'|} + \frac{(p-x)\cdot(p'-x)}{|x-p|^m|x-p'|^3}$$

and

$$(p-x)\cdot\nabla\phi(x;p,p') = -\left(|x-p| + \frac{(p-x)\cdot(p'-x)}{|x-p'|}\right),$$

we have

$$\begin{aligned}
I_m(\tau) &= (m-3)\int_D \frac{e^{-\tau\phi(x;p,p')}}{|x-p|^m|x-p'|}dx + \int_D \frac{(p-x)\cdot(p'-x)}{|x-p|^m|x-p'|^3}e^{-\tau\phi(x;p,p')}dx \\
&\quad -\tau\int_D \frac{(p-x)\cdot\nabla\phi(x;p,p')}{|x-p|^m|x-p'|}e^{-\tau\phi(x;p,p')}dx \\
&= (m-3)\int_D \frac{e^{-\tau\phi(x;p,p')}}{|x-p|^m|x-p'|}dx + \int_D \frac{(p-x)\cdot(p'-x)}{|x-p|^m|x-p'|^3}e^{-\tau\phi(x;p,p')}dx \\
&\quad +\tau\int_D \frac{e^{-\tau\phi(x;p,p')}}{|x-p|^{m-1}|x-p'|}dx + \tau\int_D \frac{(p-x)\cdot(p'-x)}{|x-p|^m|x-p'|^2}e^{-\tau\phi(x;p,p')}dx.
\end{aligned}$$

This yields

$$\begin{aligned}
I_2(\tau) + \frac{1}{\tau}I_3(\tau) &= \tau\int_D \left\{1 + \frac{(p-x)\cdot(p'-x)}{|x-p||x-p'|}\right\}\frac{e^{-\tau\phi(x;p,p')}}{|x-p||x-p'|}dx \\
&\quad + \int_D \left(\frac{1}{|x-p|} + \frac{1}{|x-p'|}\right)\frac{(p-x)\cdot(p'-x)}{|x-p|^2|x-p'|^2}e^{-\tau\phi(x;p,p')}dx \\
&\quad + \frac{1}{\tau}\int_D \frac{(p-x)\cdot(p'-x)}{|x-p|^3|x-p'|^3}e^{-\tau\phi(x;p,p')}dx.
\end{aligned} \tag{3.15}$$

Lemma 3.6. *Let D be an arbitrary nonempty bounded open set. If p and p' be arbitrary points in $\mathbf{R}^3\setminus\overline{D}$ such that $[p,p']\cap\overline{D}=\emptyset$, then*

$$C_D(p,p') \equiv \inf_{x\in D}\left\{1 + \frac{(p-x)\cdot(p'-x)}{|p-x||p'-x|}\right\} > 0. \tag{3.16}$$

Proof. It is easy to see that $0 \le C_D(p,p') \le 2$. Assume that $C_D(p,p')=0$. Since we have the identity

$$1 + \frac{(p-x)\cdot(p'-x)}{|p-x||p'-x|} = \frac{1}{2}\left|\frac{p-x}{|p-x|} + \frac{p'-x}{|p'-x|}\right|^2,\ x \ne p, p',$$

there exists a sequence $\{x_n\}$ in D such that

$$\left|\frac{p-x_n}{|p-x_n|} + \frac{p'-x_n}{|p'-x_n|}\right|^2 \longrightarrow 0. \tag{3.17}$$

Since $\overline{D}$ is compact, choosing a subsequence of $\{x_n\}$ if necessary, one may assume that $\{x_n\}$ converges to a point $y\in\overline{D}$. Since $p, p' \ne y$ by assumption, it follows from (3.17) that

$$\frac{p-y}{|p-y|} + \frac{p'-y}{|p'-y|} = 0.$$

This gives $y \in [p,p']$ and thus $[p,p'] \cap \overline{D} \neq \emptyset$. This is a contradiction.
$\square$

A combination of (3.15) and (3.16) gives

$$I_2(\tau) + \frac{1}{\tau} I_3(\tau) \geq \tau K(\tau) \int_D e^{-\tau\phi(x;p,p')} dx, \tag{3.18}$$

where

$$K(\tau) = \frac{C_D(p,p')}{d_{\partial D}(p) d_{\partial D}(p')} - \left(\frac{1}{d_{\partial D}(p)} + \frac{1}{d_{\partial D}(p')}\right) \frac{1}{d_{\partial D}(p) d_{\partial D}(p')} \frac{1}{\tau} - \frac{1}{d_{\partial D}(p)^2 d_{\partial D}(p')^2} \frac{1}{\tau^2}.$$

Lemma 3.7. *Let $p, p' \in \mathbf{R}^3$. Then, there exists a number μ such that*

$$\liminf_{\tau \longrightarrow \infty} \tau^{\mu} e^{\tau \min_{x \in \partial D} \phi(x;p,p')} \int_D e^{-\tau\phi(x;p,p')} dx > 0. \tag{3.19}$$

Proof. Let $x_0 \in \partial D$ be a point such that $\phi(x_0;p,p') = \min_{x \in \partial D} \phi(x;p,p')$. Since $|p-x| \leq |p-x_0| + |x_0-x|$ and $|p'-x| \leq |p'-x_0| + |x_0-x|$, we have

$$\phi(x;p,p') \leq \phi(x_0;p,p') + 2|x_0-x|, \ \forall x \in \mathbf{R}^3.$$

This gives

$$e^{\tau \min_{x \in \partial D} \phi(x;p,p')} \int_D e^{-\tau\phi(x;p,p')} dx \geq \int_D e^{-2\tau|x-x_0|} dx.$$

In [15] we have already known that

$$\liminf_{\tau \longrightarrow \infty} \tau^3 \int_D e^{-2\tau|x-x_0|} dx > 0.$$

Thus (3.19) is valid for $\mu = 3$.
$\square$

Now it follows from (3.14), (3.18) and (3.19) that (3.12) is valid.

4 Asymptotic behaviour of the indicator function

First we claim that

$$\int_{\mathbf{R}^3 \setminus \overline{D}} (f v_g - w_f g) dx = 2J(\tau;f,g)(1 + O(\tau^{-1/2})) + O(\tau^{-1} e^{-\tau T}). \tag{4.1}$$

This is a consequence of the following asymptotic formula and (2.10).

Proposition 4.1. *As $\tau \longrightarrow \infty$,*

$$\int_{\mathbf{R}^3 \setminus \overline{D}} (\nabla \epsilon_f^0 \cdot \nabla \epsilon_g^0 + \tau^2 \epsilon_f^0 \epsilon_g^0) dx = J(\tau;f,g)(1 + O(\tau^{-1/2})).$$

Proposition 4.1 is a direct consequence of the following lemmas.

Lemma 4.1. *Let D be an arbitrary nonempty bounded open set. If B and B' be arbitrary open balls such that $[\overline{B}\cup\overline{B}']\cap\overline{D}=\emptyset$, then there exist positive constants C and τ_0 independent of τ such that, for all $\tau\ge\tau_0$, $J(\tau;f,g)>0$ and*

$$J(\tau;f,g)\ge C\tau^2\int_D dx\int_{B\times B'}e^{-\tau\phi(x;y,y')}dydy'.$$

Proof. From (1.3) and (2.1) we have

$$J(\tau;f,g)=\left(\frac{1}{4\pi}\right)^2\int_D dx\int_{B\times B'}dydy'\frac{K_\tau(x,y,y')}{|x-y||x-y'|}e^{-\tau\phi(x;y,y')}, \tag{4.2}$$

where

$$K_\tau(x,y,y')=\left(\frac{1}{|x-y|}+\tau\right)\left(\frac{1}{|x-y'|}+\tau\right)\frac{(y-x)}{|x-y|}\cdot\frac{(y'-x)}{|x-y'|}+\tau^2.$$

Since $\overline{B}\cap\overline{D}=\overline{B}'\cap\overline{D}=\emptyset$, there exist positive constants C_1 and C_2 such that

$$K_\tau(x,y,y')\ge\tau^2\left(1+\frac{(y-x)}{|x-y|}\cdot\frac{(y'-x)}{|x-y'|}\right)-C_1\tau-C_2,\ \forall(x,y,y')\in D\times B\times B'. \tag{4.3}$$

Here we claim the following estimate:

$$C_D(B,B')\equiv\inf_{(x,y,y')\in D\times B\times B'}\left(1+\frac{y-x}{|y-x|}\cdot\frac{y'-x}{|y'-x|}\right)>0. \tag{4.4}$$

This is proved as follows.

It is easy to see that $0\le C_D(B,B')\le 2$. Assume that $C_D(B,B')=0$. Since we have the identity

$$1+\frac{y-x}{|y-x|}\cdot\frac{y'-x}{|y'-x|}=\frac{1}{2}\left|\frac{y-x}{|y-x|}+\frac{y'-x}{|y'-x|}\right|^2,\ x\ne y,y',$$

there exist sequences $\{x_n\}$ in D, $\{y_n\}$ in B and $\{y_n'\}$ in B' such that

$$\left|\frac{y_n-x_n}{|y_n-x_n|}+\frac{y_n'-x_n}{|y_n'-x_n|}\right|^2\longrightarrow 0. \tag{4.5}$$

Since $\overline{D}\times\overline{B}\times\overline{B}'$ is compact, choosing a subsequence of $\{(x_n,y_n,y_n')\}$ if necessary, one may assume that $\{(x_n,y_n,y_n')\}$ converges to a point $(x_*,y_*,y_*')\in\overline{D}\times\overline{B}\times\overline{B}'$. Since $y_*,y_*'\ne x_*$ by assumption, it follows from (4.5) that

$$\frac{y_*-x_*}{|y_*-x_*|}+\frac{y_*'-x_*}{|y_*'-x_*|}=0.$$

This gives $x_*\in\{sy_*+(1-s)y_*'\,|\,0<s<1\}$ and thus $[\overline{B}\cup\overline{B}']\cap\overline{D}\ne\emptyset$. This is a contradiction.

Thus, applying (4.4) to the right-hand side of (4.3), we obtain, for a sufficiently large $\tau_0>0$

$$\inf_{(x,y,y')\in D\times B\times B'}K_\tau(x,y,y')\ge\frac{\tau^2}{2}C_D(B,B'),\ \forall\tau\ge\tau_0'.$$

A combination of this and (4.2) ensures the validity of Lemma 4.1 with

$$C = \left(\frac{1}{4\pi}\right)^2 \frac{\tau^2 C_D(B,B')}{2\mathrm{dist}\,(D,B)\,\mathrm{dist}\,(D,B')}.$$

□

Lemma 4.2. *Assume that D is admissible and its boundary is C^3. Then, there exist positive constants C and τ_0 independent of τ such that, for all $\tau \ge \tau_0$,*

$$\left|\int_{\mathbf{R}^3\setminus\overline{D}} (\nabla\epsilon_f^0\cdot\nabla\epsilon_g^0 + \tau^2\epsilon_f^0\epsilon_g^0)dx - J(\tau;f,g)\right| \le C\tau^{3/2}\int_D dx\int_{B\times B'} e^{-\tau\phi(x;y,y')}dydy'.$$

Proof. Let $0<\delta<\delta_0$. Let $\phi=\phi_\delta$ be a smooth cut-off function, $0\le\phi(x)\le 1$, and such that: $\phi(x)=1$ if $d_{\partial D}(x)<\delta$ and $\phi(x)=0$ if $d_{\partial D}(x)>2\delta$; $|\nabla\phi(x)|\le C\delta^{-1}$; $|\nabla^2\phi(x)|\le C\delta^{-2}$.

For $*=f,g$ define $(\epsilon_*^0)^r(x)=\phi(x)\epsilon_*^0(x^r)$ for $x\in D$ and $v_*^r(x)=\phi(x)v_*(x^r)$ for $x\in\mathbf{R}^3\setminus\overline{D}$.

The Lax-Phillips reflection argument starts with the following expression:

$$\int_{\mathbf{R}^3\setminus\overline{D}} (\nabla\epsilon_f^0\cdot\nabla\epsilon_g^0 + \tau^2\epsilon_f^0\epsilon_g^0)dx = J(\tau;f,g) + \int_{\mathbf{R}^3\setminus\overline{D}} \epsilon_f^0(\triangle-\tau^2)v_g^r dx. \tag{4.6}$$

In the proof the following relationship between v_*^r and v_* and the boundary condition for ϵ_*^0 are essential:

$$\frac{\partial v_*^r}{\partial\nu} = -\frac{\partial v_*}{\partial\nu}\ \text{on}\,\partial D,$$

$$v_*^r = v_* = -\epsilon_*^0\ \text{on}\,\partial D.$$

Another device is the following differential identity which is a consequence of (4.15) in [18](see also Appendix 1 in [12]):

$$(\triangle-\tau^2)(v_g^r) = \phi(x)\sum_{i,j} a_{ij}(x)(\partial_i\partial_j v_g)(x^r) + \sum_j b_j(x)(\partial_j v_g)(x^r) + (\triangle\phi)(x)v_g(x^r).$$

where $a_{ij}(x)$, $i,j=1,2,3$ are C^1 in a neighbourhood of ∂D, independent of ϕ and v_g and satisfy

$$\exists C>0\,\forall x\in\mathbf{R}^3\ \ \phi(x)|a_{ij}(x)|\le C d_{\partial D}(x); \tag{4.7}$$

each $b_j(x)$ has the form

$$b_j(x) = \sum_{jk} b_{jk}(x)\partial_k\phi(x) + d_j(x)\phi(x) \tag{4.8}$$

with $b_{jk}(x)$ and $d_j(x)$ which are C^1 and C^0 in a neighbourhood of ∂D, respectively and independent of ϕ and v_g.

This together with the change of variables $x = y^r$ yields

$$\int_{\mathbf{R}^3\setminus\overline{D}} \epsilon_f^0(x)(\triangle - \tau^2)v_g^r(x)dx$$

$$= \int_{\mathbf{R}^3\setminus\overline{D}} \epsilon_f^0(x)\left\{\phi(x)\sum_{i,j} a_{ij}(x)(\partial_i\partial_j v_g)(x^r) + \sum_j b_j(x)(\partial_j v_g)(x^r) + (\triangle\phi)(x)v_g(x^r)\right\}dx$$

$$= \int_D \epsilon_f^0(y^r)\left\{\phi(y^r)\sum_{i,j} a_{ij}(y^r)(\partial_i\partial_j v_g)(y) + \sum_j b_j(y^r)(\partial_j v_g)(y) + (\triangle\phi)(y^r)v_g(y)\right\}J(y)dy, \tag{4.9}$$

where $J(y)$ denotes the Jacobian.

Hereafter, we give an estimation for each term in the right-hand side of (4.9) pointwisely, without making use of integration by parts further. This idea is exactly same as the proof of Lemma 3.3 in [18]. This is different from the back-scattering case, see also the proof of Lemma 4.2 in [18] and Appendix 1 in [12] for the comparison.

Since D is admissible we have, for all $y \in D$ with $d_{\partial D}(y) < \delta'$ and all $\tau > \tau_0$

$$|\epsilon_f^0(y^r)| \le C\int_B e^{-\tau|y-x|}dx, \tag{4.10}$$

where C is independent of y and τ.

It is easy to see that, for all $y \in D$,

$$\tau^{-2}|\partial_i\partial_j v_g(y)| + \tau^{-1}|\partial_i v_g(y)| + |v_g(y)| \le C\int_{B'} e^{-\tau|y-x'|}dx'.$$

From this, (4.10), (4.7) and (4.8) and the choice of ϕ we obtain, for all $y \in D$,

$$\begin{array}{l}\displaystyle\left|\epsilon_f^0(y^r)\left\{\phi(y^r)\sum_{i,j} a_{ij}(y^r)(\partial_i\partial_j v_g)(y) + \sum_j b_j(y^r)(\partial_j v_g)(y) + (\triangle\phi)(y^r)v_g(y)\right\}\right| \\ \displaystyle\le C(\delta\tau^2 + \delta^{-1}\tau + \delta^{-2})\int_{B\times B'} e^{-\tau\phi(y;x,x')}dxdx'.\end{array} \tag{4.11}$$

Choosing $\delta = \tau^{-1/2}$, we have $\delta\tau^2 + \delta^{-1}\tau + \delta^{-2} = O(\tau^{3/2})$ as $\tau \longrightarrow \infty$. Now from (4.9) and (4.11) we obtain the desired conclusion of Lemma 4.2.

□

Thus everything is reduced to studying the asymptotic behaviour of $J(\tau; f, g)$ as $\tau \longrightarrow \infty$. For this purpose we employ the asymptotic formula (3.13). Note that in Section 3 we made use of the formula to give a lower estimate of $J(\tau; f, g)$. Here using the formula, we determine its leading term as $\tau \longrightarrow \infty$.

From (3.13) we see that the asymptotic behaviour of the following integral is the key:

$$\int_{\partial D} \frac{(p-x)\cdot\nu_x}{|x-p|^2|x-p'|}\left(1 + \frac{1}{\tau|x-p|}\right)e^{-\tau\phi(x;p,p')}dS_x.$$

Proposition 4.2. *Assume that $\Lambda_{\partial D}(p,p')$ is finite and*

$$det(S_q(E_c(p,p')) - S_q(\partial D)) > 0 \;\forall q \in \Lambda_{\partial D}(p,p').$$

Then, we have

$$\lim_{\tau\longrightarrow\infty}\tau e^{\tau\phi(q;p,p')}\int_{\partial D}\frac{(p-x)\cdot\nu_x}{|x-p|^2|x-p'|}\left(1+\frac{1}{\tau|x-p|}\right)e^{-\tau\phi(x;p,p')}dS_x$$

$$=\sum_{q\in\Lambda_{\partial D}(p,p')}\frac{\pi}{|q-p||q-p'|}\frac{1}{\sqrt{det\,(S_q(E_c(p,p'))-S_q(\partial D))}}.$$

Theorem 1.3 is a direct consequence of this together with (3.13) and (4.1).

In the following subsection, we describe the proof of Proposition 4.2.

4.1 Proof of Proposition 4.2.

We employ the Laplace method and so one has to compute the Hessian of the real phase function $\partial D\ni x\longmapsto\phi(x;p,p')$ at all the points on ∂D where it takes the minimum value.

Let $q\in\Lambda_{\partial D}(p,p')$. One can choose a local coordinates system $\sigma=(\sigma_1,\sigma_2)$ around q on ∂D in such a way that $x\in\partial D$ around q has the form

$$x=x_q(\sigma)=q+\sigma_1\boldsymbol{e}_1+\sigma_2\boldsymbol{e}_2+f(\sigma)\nu_q,\ |\sigma|<\delta,$$

where δ is a sufficiently small positive number independent of q; $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are two unit tangent vectors at q to ∂D which are perpendicular to each other and $\boldsymbol{e}_1\times\boldsymbol{e}_2=\nu_q$; $f=f_q\in C^3_0(\mathbf{R}^3)$ and satisfies $f(0)=0$, $\nabla f(0)=0$; ν_x takes the form

$$\nu_x=\frac{-f_1\boldsymbol{e}_1-f_2\boldsymbol{e}_2+\nu_q}{\sqrt{1+f_1^2+f_2^2}},$$

where $f_1=\partial f/\partial\sigma_1$ and $f_2=\partial f/\partial\sigma_2$; $dS=\sqrt{1+f_1^2+f_2^2}d\sigma$.

We have, for $z=p,p'$ and $j=1,2$,

$$\frac{\partial}{\partial\sigma_j}|x_q(\sigma)-z|=\frac{1}{|x_q(\sigma)-z|}\left\{(x_q(\sigma)-z)\cdot\boldsymbol{e}_j+f_j(\sigma)(x_q(\sigma)-z)\cdot\nu_q\right\}\tag{4.12}$$

and thus

$$\frac{\partial^2}{\partial\sigma_k\partial\sigma_j}|x_q(\sigma)-z|$$

$$=-\frac{\{(x_q(\sigma)-z)\cdot\boldsymbol{e}_k+f_k(\sigma)(x_q(\sigma)-z)\cdot\nu_q\}\{(x_q(\sigma)-z)\cdot\boldsymbol{e}_j+f_j(\sigma)(x_q(\sigma)-z)\cdot\nu_q\}}{|x_q(\sigma)-z|^3}$$

$$+\frac{1}{|x_q(\sigma)-z|}\left(\delta_{kj}+\frac{\partial^2 f}{\partial\sigma_k\partial\sigma_j}(\sigma)(x_q(\sigma)-z)\cdot\nu_q+f_j(\sigma)f_k(\sigma)\right).$$

This yields

$$\frac{\partial^2}{\partial\sigma_k\partial\sigma_j}\phi(x_q(\sigma);p,p')|_{\sigma=0}=\lambda(q;p,p')\delta_{kj}-a_{kj}(q;p,p'),\tag{4.13}$$

where

$$\lambda(q;p,p') = \frac{1}{|p-q|} + \frac{1}{|p'-q|},$$

$$a_{kj}(q;p,p') = \frac{1}{|p-q|}\boldsymbol{A}\cdot\boldsymbol{e}_k\boldsymbol{A}\cdot\boldsymbol{e}_j + \frac{1}{|p'-q|}\boldsymbol{A}'\cdot\boldsymbol{e}_k\boldsymbol{A}'\cdot\boldsymbol{e}_j - \frac{\partial^2 f}{\partial\sigma_k\partial\sigma_j}(0)\,(\boldsymbol{A}\cdot\nu_q + \boldsymbol{A}'\cdot\nu_q)$$

and

$$\boldsymbol{A} = \boldsymbol{A}_q(p),\ \ \boldsymbol{A}' = \boldsymbol{A}_q(p').$$

Note that $\lambda(q;p,p') = \lambda(q;p',p)$ and $a_{kj}(q;p,p') = a_{kj}(q;p',p)$.

The following lemma corresponds to Snell's law in geometrical optics.

Lemma 4.3. *Let p and p' be arbitrary points in $\mathbf{R}^3 \setminus \partial D$ such that $[p,p'] \cap \partial D = \emptyset$. Let $q \in \Lambda_{\partial D}(p,p')$. Then, we have: (i) the vector $\boldsymbol{A}+\boldsymbol{A}'$ is parallel to ν_q; (ii) $\boldsymbol{A}\cdot\nu_q = \boldsymbol{A}'\cdot\nu_q \neq 0$. In addition, if p and p' are in $\mathbf{R}^3 \setminus \overline{D}$, then we have*

$$\nu_q = -\frac{1}{\sqrt{2(1+\boldsymbol{A}\cdot\boldsymbol{A}')}}(\boldsymbol{A}+\boldsymbol{A}'), \tag{4.14}$$

that is, the unit inward normal to $E_c(p,p')$ at q with $c = \phi(q;p,p')$ (see (A.17) in Appendix) coincides with the unit outward normal to ∂D at the same point.

Proof. Since the function $\sigma \mapsto \phi(x_q(\sigma);p,p')$ takes its minimum at $\sigma = 0$, it follows from (4.12) that

$$\begin{aligned}\boldsymbol{A}\cdot\boldsymbol{e}_1 + \boldsymbol{A}'\cdot\boldsymbol{e}_1 = 0\\ \boldsymbol{A}\cdot\boldsymbol{e}_2 + \boldsymbol{A}'\cdot\boldsymbol{e}_2 = 0.\end{aligned} \tag{4.15}$$

Write

$$\begin{aligned}\boldsymbol{A} = \alpha\boldsymbol{e}_1 + \beta\boldsymbol{e}_2 + \gamma\nu_q,\\ \boldsymbol{A}' = \alpha'\boldsymbol{e}_1 + \beta'\boldsymbol{e}_2 + \gamma'\nu_q.\end{aligned} \tag{4.16}$$

(4.15) is equivalent to $\alpha + \alpha' = 0$ and $\beta + \beta' = 0$. Then we have $\gamma^2 = \gamma'^2$, that is, $\gamma = \gamma'$ or $\gamma = -\gamma'$. Assume that $\gamma = -\gamma'$. Then from (4.16) we have $\boldsymbol{A}+\boldsymbol{A}' = 0$, that is,

$$\frac{q-p}{|q-p|} + \frac{q-p'}{|q-p'|} = 0.$$

This means that $q \in [p,p']$ and contradicts the condition $[p,p'] \cap \partial D = \emptyset$. Thus $\gamma = \gamma'$ and we have

$$\boldsymbol{A}+\boldsymbol{A}' = 2\gamma\nu_q \tag{4.17}$$

and

$$\gamma = \pm\frac{\sqrt{1+\boldsymbol{A}\cdot\boldsymbol{A}'}}{\sqrt{2}}.$$

Since $|\boldsymbol{A}+\boldsymbol{A}'|^2 = 2(1+\boldsymbol{A}\cdot\boldsymbol{A}')$, from the argument above we have $1+\boldsymbol{A}\cdot\boldsymbol{A}' > 0$. Thus from (4.17), one gets $\boldsymbol{A}\cdot\nu_q \neq 0$ and $\boldsymbol{A}'\cdot\nu_q \neq 0$.

Note that if p and p' are in $\mathbf{R}^3 \setminus \overline{D}$, γ has to be negative. The reason is the following. Assume that $\gamma > 0$. Then $-(\boldsymbol{A}+\boldsymbol{A}')$ is directed to $-\nu_q$. Since ∂D is C^2, one can find a sufficiently small $s > 0$ such that $q' \equiv q - s(\boldsymbol{A}+\boldsymbol{A}') \in D$. Since $p \in \mathbf{R}^3 \setminus \overline{D}$, one can find a

point $p'' \in \partial D$ on the segment with endpoints q' and p. Note that $-(\boldsymbol{A}+\boldsymbol{A}')$ is directed to the unit inward normal to $E_c(p,p')$. This together with the condition $c > |p-p'|$ ensures that both q' and p are in the domain enclosed by $E_c(p,p')$ and thus by the convexity of the domain, we have $\phi(p'';p,p') < c$. However, since $p'' \in \partial D$, we have $\phi(p'';p,p') \geq c$. This is a contradiction. Therefore one gets $\gamma < 0$ and now it is easy to see that all the conclusions are valid.
□

It follows from (i) in Lemma 4.3 that

$$\boldsymbol{A}\cdot\boldsymbol{e}_k\boldsymbol{A}\cdot\boldsymbol{e}_j = \boldsymbol{A}'\cdot\boldsymbol{e}_k\boldsymbol{A}'\cdot\boldsymbol{e}_j.$$

Thus, we have

$$\frac{1}{|p-q|}\boldsymbol{A}\cdot\boldsymbol{e}_k\boldsymbol{A}\cdot\boldsymbol{e}_j$$

$$=\left(\frac{1}{|p-q|}+\frac{1}{|p'-q|}\right)\boldsymbol{A}\cdot\boldsymbol{e}_k\boldsymbol{A}\cdot\boldsymbol{e}_j - \frac{1}{|p'-q|}\boldsymbol{A}\cdot\boldsymbol{e}_k\boldsymbol{A}\cdot\boldsymbol{e}_j$$

$$=\lambda(q;p,p')\boldsymbol{A}\cdot\boldsymbol{e}_k\boldsymbol{A}\cdot\boldsymbol{e}_j - \frac{1}{|p'-q|}\boldsymbol{A}'\cdot\boldsymbol{e}_k\boldsymbol{A}'\cdot\boldsymbol{e}_j$$

and hence

$$\frac{1}{|p-q|}\boldsymbol{A}\cdot\boldsymbol{e}_k\boldsymbol{A}\cdot\boldsymbol{e}_j + \frac{1}{|p'-q|}\boldsymbol{A}'\cdot\boldsymbol{e}_k\boldsymbol{A}'\cdot\boldsymbol{e}_j = \lambda(q;p,p')\boldsymbol{A}\cdot\boldsymbol{e}_k\boldsymbol{A}\cdot\boldsymbol{e}_j.$$

Changing the role of p and p', we also have

$$\frac{1}{|p-q|}\boldsymbol{A}\cdot\boldsymbol{e}_k\boldsymbol{A}\cdot\boldsymbol{e}_j + \frac{1}{|p'-q|}\boldsymbol{A}'\cdot\boldsymbol{e}_k\boldsymbol{A}'\cdot\boldsymbol{e}_j = \lambda(q;p,p')\boldsymbol{A}'\cdot\boldsymbol{e}_k\boldsymbol{A}'\cdot\boldsymbol{e}_j.$$

From these and (4.14), we obtain another expression for $a_{jk}(q:p,p')$:

$$a_{jk}(q;p,p') = \frac{\lambda(q;p,p')}{2}\left(\boldsymbol{A}\cdot\boldsymbol{e}_k\boldsymbol{A}\cdot\boldsymbol{e}_j + \boldsymbol{A}'\cdot\boldsymbol{e}_k\boldsymbol{A}'\cdot\boldsymbol{e}_j\right) + \sqrt{2(1+\boldsymbol{A}\cdot\boldsymbol{A}')}\frac{\partial^2 f}{\partial\sigma_k\partial\sigma_j}(0).$$

This together with (4.13) implies that

$$\begin{aligned}\frac{\partial^2}{\partial\sigma_k\partial\sigma_j}\phi(x_q(\sigma);p,p')|_{\sigma=0} &= \sqrt{2(1+\boldsymbol{A}\cdot\boldsymbol{A}')}\\ &\times\left\{\frac{\lambda(q;p,p')}{\sqrt{2(1+\boldsymbol{A}\cdot\boldsymbol{A}')}}\left(\delta_{jk}-\frac{1}{2}(\boldsymbol{A}\cdot\boldsymbol{e}_k\boldsymbol{A}\cdot\boldsymbol{e}_j + \boldsymbol{A}'\cdot\boldsymbol{e}_k\boldsymbol{A}'\cdot\boldsymbol{e}_j)\right) - \frac{\partial^2 f}{\partial\sigma_k\sigma_j}(0)\right\}.\end{aligned} \tag{4.18}$$

From (A.2) we have

$$\begin{pmatrix}\boldsymbol{e}_1^T\\ \\ \boldsymbol{e}_2^T\end{pmatrix} S_q(E_c(p,p'))\begin{pmatrix}\boldsymbol{e}_1 & \boldsymbol{e}_2\end{pmatrix}$$

$$=\frac{\lambda(q;p,p')}{\sqrt{2(1+\boldsymbol{A}\cdot\boldsymbol{A}')}}\left(\delta_{jk}-\frac{1}{2}(\boldsymbol{A}\cdot\boldsymbol{e}_k\boldsymbol{A}\cdot\boldsymbol{e}_j+\boldsymbol{A}'\cdot\boldsymbol{e}_k\boldsymbol{A}'\cdot\boldsymbol{e}_j)\right)_{j\downarrow 1,2;k\rightarrow 1,2}$$

and we know

$$\begin{pmatrix} \boldsymbol{e}_1^T \\ \\ \boldsymbol{e}_2^T \end{pmatrix} S_q(\partial D) \begin{pmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 \end{pmatrix} = \left(\frac{\partial^2 f}{\partial\sigma_j \partial\sigma_k}(0)\right)_{j\downarrow 1,2;k\rightarrow 1,2}. \tag{4.19}$$

Thus we obtain the following formula which gives the geometrical meaning of the Hessian of $\phi(x_q(\sigma);p,p')$ at $\sigma=0$.

Lemma 4.4. *Let p and p' be in $\mathbf{R}^3\setminus\overline{D}$ such that $[p,p']\cap\partial D=\emptyset$. Let $q\in\Lambda_{\partial D}(p,p')$ and $c=\phi(q;p,p')$. Then, $c>|p-p'|$ and we have*

$$\nabla_\sigma^2\phi(x_q(\sigma);p,p')|_{\sigma=0}=\sqrt{2(1+\boldsymbol{A}\cdot\boldsymbol{A}')}\begin{pmatrix} \boldsymbol{e}_1^T \\ \\ \boldsymbol{e}_2^T \end{pmatrix}(S_q(E_c(p,p'))-S_q(\partial D))\begin{pmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 \end{pmatrix}. \tag{4.20}$$

Since $\phi(x_q(\sigma);p,p')$ takes the minimum at $\sigma=0$, from (4.20) one concludes that, for all tangent vectors $\boldsymbol{v}$ at q

$$(S_q(E_c(p,p'))-S_q(\partial D))\boldsymbol{v}\cdot\boldsymbol{v}\ge 0. \tag{4.21}$$

Thus if (1.10) is satisfied, then from (4.21) one knows that $S_q(E_c(p,p'))-S_q(\partial D)$ is positive definite on the tangent space at q and from (4.20)

$$\det\nabla_\sigma^2\phi(x_q(\sigma);p,p')|_{\sigma=0}=2(1+\boldsymbol{A}\cdot\boldsymbol{A}')\det\left(S_q(E_c(p,p'))-S_q(\partial D)\right)>0.$$

And also from (4.14) we have

$$(p-q)\cdot\nu_q=\frac{|q-p|}{\sqrt{2}}\sqrt{1+\boldsymbol{A}\cdot\boldsymbol{A}'}.$$

Now Proposition 4.2 is a direct consequence of the Laplace method [2].

5 Proof of Theorems 1.2 and 1.4

5.1 Extracting the first reflector: proof of Theorem 1.2

Let $c>|p-p'|$. Theorem 1.2 is based on the following proposition which gives a characterization of a first reflection point q between p and p' in terms of the minimum length of the broken path connecting p to q and q to $p'+s(q-p')/|q-p'|$ with a fixed small $s>0$.

Proposition 5.1. *Fix $0<s<\eta'$. Let $c=\min_{x\in\partial D}\phi(x;p,p')$. We have:*
(i) if $p'+s(\omega;p,p',c)\omega$ belongs to ∂D, then

$$\min_{x\in\partial D}\phi(x;p,p'+s\omega)=c-s;$$

(ii) if $p'+s(\omega;p,p',c)\omega$ does not belong to ∂D, then

$$\min_{x\in\partial D}\phi(x;p,p'+s\omega)>c-s.$$

Thus, one has the following characterization of the first reflector:

$$\Lambda_{\partial D}(p,p') = \{p' + s(\omega;p,p',c)\omega \mid \min_{x\in\partial D}\phi(x;p,p'+s\omega) = c-s,\ \omega\in S^2\}.$$

Proof. Set $p''(\omega) = p' + s\omega$. Let $x \in \partial D$. We have

$$\begin{aligned}\phi(x;p,p''(\omega)) &= |p-x| + |x-p''(\omega)|\\ &= |p-x| + |(x-p') + (p'-p''(\omega))| \qquad (5.1)\\ &\ge |p-x| + |x-p'| - |p'-p''(\omega)| = \phi(x;p,p') - s.\end{aligned}$$

This gives

$$\min_{x\in\partial D}\phi(x;p,p''(\omega)) \ge c-s. \tag{5.2}$$

Now we describe the proof of (i). Noting that $s < s(\omega;p,p',c)$ and $p' + s(\omega;p,p',c)\omega \in E_c(p,p')$, we have

$$\begin{aligned}&\phi(p'+s(\omega;p,p',c)\omega;p,p''(\omega))\\ &= |p-(p'+s(\omega;p,p',c)\omega)| + |(p'+s(\omega;p,p',c)\omega) - p''(\omega)|\\ &= |p-(p'+s(\omega;p,p',c)\omega)| + (s(\omega;p,p',c)-s)\\ &= |p-(p'+s(\omega;p,p',c)\omega)| + |(p'+s(\omega;p,p',c)\omega) - p'| - s\\ &= \phi(p'+s(\omega;p,p',c)\omega);p,p') - s = c-s.\end{aligned}$$

Thus if $p' + s(\omega;p,p',c)\omega$ belongs to ∂D, then the inequality in (5.2)

$$\min_{x\in\partial D}\phi(x;p,p''(\omega)) > c-s$$

never occurs. Thus it must hold that $\min_{x\in\partial D}\phi(x;p,p''(\omega)) = c-s$.

Since we have always (5.2), the statement of (ii) is equivalent to the one that: if $\min_{x\in\partial D}\phi(x;p,p''(\omega)) = c-s$, then $p'+s(\omega;p,p',c)\omega \in \partial D$.

Assume that $\min_{x\in\partial D}\phi(x;p,p''(\omega)) = c-s$. Choose a point $x \in \partial D$ such that $\phi(x;p,p''(\omega)) = c-s$. Then, from (5.1) we have $c \ge \phi(x;p,p')$. Since $c = \min_{x\in\partial D}\phi(x;p,p')$, from this we obtain $\phi(x;p,p') = c$. Thus we have $\phi(x;p,p''(\omega)) = \phi(x;p,p') - s$. This is equivalent to

$$|x-p''(\omega)| = |x-p'| - s. \tag{5.3}$$

Since x is outside the open ball B'' centered at p' with radius s, one can find the unique point x' on $\partial B''$ such that $|x-x'| = \min_{y\in\partial B''}|y-x|$. Since we have $|x-p'| = |x-x'|+s$, from (5.3) we obtain $|x-p''(\omega)| = |x-x'|$. Since $p''(\omega) \in \partial B''$, it must hold that $p''(\omega) = x'$ and thus $x = p' + s(\omega;p,p',c)\omega$. This gives $p'+s(\omega;p,p',c)\omega \in \partial D$.
□

Now we are ready to describe the proof of Theorem 1.2.

In what follows we denote the open ball centered at a point z and with radius ρ by $B_\rho(z)$. Since $B_{\eta'-s}(p'+s\omega)$ is contained in B', from u_f on $B'\times]0,T[$ with $f=\chi_B$, one

gets u_f on $B_{\eta'-s}(p'+s\omega)\times\,]0,\,T[$. By Theorem 1.1 we obtain $\inf_{x\in\partial D}\phi(x;p,p'+s\omega)$ for each $\omega\in S^2$. Thus, from Proposition 5.1 we obtain $\Lambda_{\partial D}(p,p')$ itself. From formula (4.14) one gets ν_q at given $q\in\Lambda_{\partial D}(p,p')$.

Thus one can complectly determine the first reflectors between p and p' using the bistatic data u_f on $B'\times\,]0,\,T[$ for $f=\chi_B$ and sufficiently large and *fixed* T. In particular, note that B is *fixed*.

Remark 5.1. We summarize how to detect the points in $\Lambda_{\partial D}(p,p')\subset E_c(p,p')$.

Step 1. Compute w_f on B' with $f=\chi_B$ from the data u_f on $B'\times\,]0,\,T[$.

Step 2. Fix s with $0<s<\eta'$.

Step 3. Choose a direction $\omega\in S^2$.

Step 4. Compute the following integral with $g=\chi_{B_{\eta'-s}(p'+s\omega)}$:

$$\int_B v_g dx-\int_{B_{\eta'-s}(p'+s\omega)} w_f dx.$$

Step 5. Compute the following quantity:

$$-\lim_{\tau\longrightarrow\infty}\frac{1}{\tau}\log\left(\int_B v_g dx-\int_{B_{\eta'-s}(p'+s\omega)} w_f dx\right)-\eta-\eta'+s.$$

Step 6. If the computed quantity in Step 5 is equal to $c-s$, then $p'+s(\omega;p,p',c)\omega\in\Lambda_{\partial D}(p,p')$. If not so, then choose another ω and go to Step 4.

5.2 Extracting the geometry of ∂D at a first reflection point: proof of Theorem 1.4

In this subsection, we consider the case when a point $q\in\Lambda_{\partial D}(p,p')$ is *known*. We use the notation $\boldsymbol{A}$ and $\boldsymbol{A}'$ instead of $\boldsymbol{A}_q(p)$ and $\boldsymbol{A}_q(p')$, respectively for simplicity of description. The aim of this subsection is to extract the geometry of ∂D at q. The main idea is to replace p' in $\Lambda_{\partial D}(p,p')$ with $p'+s\boldsymbol{A}'$ with a small $s>0$. The advantage of this idea is described in the following proposition.

Proposition 5.2. *Let $c=\min_{x\in\partial D}\phi(x;p,p')$ and satisfy $c>|p-p'|$. Fix $0<s<\eta'$. If $q\in\Lambda_{\partial D}(p,p')$, then $\Lambda_{\partial D}\left(p,p'+s\boldsymbol{A}'\right)=\{q\}$. Moreover, if $s<\frac{1}{2}(c-|p-p'|)$, then the operator $S_q\left(E_{c-s}\left(p,p'+s\boldsymbol{A}'\right)\right)-S_q(\partial D)$ is positive definite on the common tangent space $T_q(\partial D)=T_q(E_{c-s}(p,p'+s\boldsymbol{A}'))=T_q(E_c(p,p'))$ at q.*

Proof. From the definition of $\Lambda_{\partial D}(p,p')$, $q\in\partial D$ and one can write

$$q=p'+s(\boldsymbol{A}';p,p',c)\boldsymbol{A}'.$$

Then, we have $\phi(q;p,p'')=c-s$, where $p''=p'+s\boldsymbol{A}'$. Since $\phi(x;p,p'')\ge c-s$ for all $x\in\partial D$, this means $c-s=\min_{x\in\partial D}\phi(x;p,p'')$ and thus

$$q\in\Lambda_{\partial D}\left(p,p'+s\boldsymbol{A}'\right).$$

Next let

$$x\in\Lambda_{\partial D}\left(p,p'+s\boldsymbol{A}'\right).$$

We have $\phi(x;p,p'') = \min_{x\in\partial D}\phi(x;p,p'') = c-s$. Then we have

$$
\begin{aligned}
c-s=\phi(x;p,p'')&=|p-x|+|x-p''|\\
&=|p-x|+|(x-p')+(p'-p'')|\\
&\ge |p-x|+|x-p'|-|p'-p''|=\phi(x;p,p')-s.
\end{aligned}
$$

Thus $\phi(x;p,p')\le c$ and hence $c=\phi(x;p,p')$. From these we have $\phi(x;p,p')-s=\phi(x;p,p'')$. This is equivalent to

$$
|x-p''|=|x-p'|-s. \tag{5.4}
$$

Since x is outside the open ball B'' centered at p' with radius s, one can find the unique point x' on $\partial B''$ such that $|x-x'|=\min_{y\in\partial B''}|y-x|$. Since we have $|x-p'|=|x-x'|+s$, from (5.4) we obtain $|x-p''|=|x-x'|$. Since $p''\in\partial B''$, it must hold that $p''=x'$ and thus $x=p'+s(\boldsymbol{A}';p,p',c)\boldsymbol{A}'=q$.

The last statement is based on the fact that the eigenvectors for both shape operators are common and $\lambda'>\lambda$, where

$$
\lambda=\lambda(q;p,p')=\frac{1}{|q-p|}+\frac{1}{|q-p'|},\ \ \lambda'=\lambda(q;p,p'+s\boldsymbol{A}')=\frac{1}{|q-p|}+\frac{1}{|q-p'|-s}.
$$

See Appendix for these. Thus one concludes that the operator $S_q\left(E_{c-s}\left(p,p'+s\boldsymbol{A}'\right)\right)-S_q(E_c(p,p'))$ is positive definite. Since $S_q(E_c(p,p'))-S_q(\partial D)\ge 0$, from these one gets the desired conclusion. Note that the condition $s<\frac{1}{2}(c-|p-p'|)$ is just for ensuring that $|p-(p'+s\boldsymbol{A}')|<c-s$.
$\square$

We have

$$
\min_{x\in\partial D,\,y\in\partial B,\,y'\in\partial B_{\eta'-s}(p'+s\boldsymbol{A}')}\phi(x;y,y')=\min_{x\in\partial D}\phi(x;p,p'+s\boldsymbol{A}')-(\eta+\eta'-s)
$$

and, by (i) of Proposition 5.1, $\min_{x\in\partial D}\phi(x;p,p'+s\boldsymbol{A}')=c-s$ if $q\in\Lambda_{\partial D}(p,p')$.

Thus the condition

$$
T>\min_{x\in\partial D,\,y\in\partial B,\,y'\in\partial B_{\eta'-s}(p'+s\boldsymbol{A}')}\phi(x;y,y')
$$

is equivalent to (1.5). Therefore the following proposition is a direct consequence of Theorem 1.3 together with Proposition 5.2.

Proposition 5.3. *Fix $0<s<\min\left(\eta',(c-|p-p'|)/2\right)$. Let B and B' satisfy $[\overline{B}\cup\overline{B}']\cap\overline{D}=\emptyset$. Let $f=\chi_B$ and $g=\chi_{B_{\eta'-s}(p'+s\boldsymbol{A}')}$. Let T satisfy (1.5). Let $q\in\Lambda_{\partial D}(p,p')$ and set $c=\phi(q;p,p')$.*

If D is admissible and ∂D is C^3, then we have

$$
\begin{aligned}
&\lim_{\tau\longrightarrow\infty}\tau^4e^{\tau(c-\eta-\eta')}\left(\int_B v_g dx-\int_{B_{\eta'-s}(p'+s\boldsymbol{A}')}w_f dx\right)\\
&=\frac{\pi}{2}\left(\frac{\eta}{|q-p|}\right)\cdot\left(\frac{\eta'-s}{|q-p'|-s}\right)\cdot\frac{1}{\sqrt{det\left(S_q(E_{c-s}(p,p'+s\boldsymbol{A}'))-S_q(\partial D)\right)}}.
\end{aligned}
\tag{5.5}
$$

It is quite interesting to know the quantities contained in $\det(S_q(E_{c-s}(p,p'+s\boldsymbol{A}'))-S_q(\partial D))$. The following lemma whose proof is given in Appendix clarifies them.

Lemma 5.1. *Let $0\le s<\min(\eta',(c-|p-p'|)/2)$. Let $q\in\Lambda_{\partial D}(p,p')$ and set $c=\phi(q;p,p')$. One has*

$$\det(S_q(E_{c-s}(p,p'+s\boldsymbol{A}'))-S_q(\partial D))=\frac{\lambda(q;p,p'+s\boldsymbol{A}')^2}{4}$$

$$-\sqrt{\frac{2}{1+\boldsymbol{A}\cdot\boldsymbol{A}'}}\lambda(q;p,p'+s\boldsymbol{A}')\left(H_{\partial D}(q)-\frac{S_q(\partial D)(\boldsymbol{A}\times\boldsymbol{A}')\cdot(\boldsymbol{A}\times\boldsymbol{A}')}{2(1+\boldsymbol{A}\cdot\boldsymbol{A}')}\right)+K_{\partial D}(q). \tag{5.6}$$

Now we are ready to describe the proof of Theorem 1.4.

Choose $0<s_1<s_2<\min(\eta',(c-|p-p'|)/2)$. Let $s=s_1,s_2$. By (5.5) in Proposition 5.3, one gets $\det(S_q(E_{c-s}(p,p'))-S_q(\partial D))$ for $s=s_1,s_2$. From (5.6) we obtain the system

$$\begin{pmatrix} -\sqrt{\frac{2}{1+\boldsymbol{A}\cdot\boldsymbol{A}'}}\lambda(q;p,p'+s_1\boldsymbol{A}') & 1\\[2ex] -\sqrt{\frac{2}{1+\boldsymbol{A}\cdot\boldsymbol{A}'}}\lambda(q;p,p'+s_2\boldsymbol{A}') & 1\end{pmatrix}\boldsymbol{X}=\boldsymbol{F}, \tag{5.7}$$

where

$$\boldsymbol{X}=\begin{pmatrix} H_{\partial D}(q)-\frac{S_q(\partial D)(\boldsymbol{A}\times\boldsymbol{A}')\cdot(\boldsymbol{A}\times\boldsymbol{A}')}{2(1+\boldsymbol{A}\cdot\boldsymbol{A}')}\\[2ex] K_{\partial D}(q)\end{pmatrix}$$

and

$$\boldsymbol{F}=\begin{pmatrix} \det(S_q(E_{c-s_1}(p,p'+s_1\boldsymbol{A}'))-S_q(\partial D))-\frac{\lambda(q;p,p'+s_1\boldsymbol{A}')^2}{4}\\[2ex] \det(S_q(E_{c-s_2}(p,p'+s_2\boldsymbol{A}'))-S_q(\partial D))-\frac{\lambda(q;p,p'+s_2\boldsymbol{A}')^2}{4}\end{pmatrix}.$$

Since $\lambda(q;p,p'+s_1\boldsymbol{A}')<\lambda(q;p,p'+s_2\boldsymbol{A}')$, (5.7) is uniquely solvable with respect to $\boldsymbol{X}$.

This completes the proof of Theorem 1.4.

Remark 5.2. It follows from (5.6) that

$$\det(S_q(E_c(p,p'))-S_q(\partial D))-P_{\partial D}(\mu;q)$$

$$=-\mu\left\{\left(\sqrt{\frac{2}{1+\boldsymbol{A}\cdot\boldsymbol{A}'}}-1\right)2H_{\partial D}(q)-\sqrt{\frac{2}{1+\boldsymbol{A}\cdot\boldsymbol{A}'}}\frac{S_q(\partial D)(\boldsymbol{A}\times\boldsymbol{A}')\cdot(\boldsymbol{A}\times\boldsymbol{A}')}{(1+\boldsymbol{A}\cdot\boldsymbol{A}')}\right\},$$

where

$$\mu=\frac{\lambda(q;p,p')}{2}=\frac{1}{2}\left(\frac{1}{|q-p|}+\frac{1}{|q-p'|}\right).$$

Since this right-hand side has a bound $O(|\boldsymbol{A}\times\boldsymbol{A}'|^2)$, this formula indicates an effect of the bistatic data on $\det(S_q(E_c(p,p'))-S_q(\partial D))$.

6 Summary and some of open problems

This paper is concerned with an inverse obstacle problem which employs the dynamical scattering data of acoustic wave over a finite time interval. The unknown obstacle D is assumed to be *sound-soft* one. The governing equation of the wave is given by the classical wave equation. The wave is generated by the initial data which is a characteristic function of an open ball B centered at p and observed over a finite time interval on a different ball B' centered at p'. It is assumed that $[\overline{B}\cup\overline{B'}]\cap\overline{D}=\emptyset$. The observed data are the so-called *bistatic data*. This is a simple mathematical model of the data collection process using an acoustic wave/electromagnetic wave such as, bistatic active *sonar*, *radar*, etc. This paper aims at developing an enclosure method which employs the bistatic data.

It is shown that from the data with some additional assumptions on the lower bound of T one can extract:

(i) the *first arrival time* in the geometrical optics sense, that is, the shortest length of the broken paths connecting p to a point $q\in\partial D$ and q to p;

(ii) the *first reflection points* between p and p', that is, all the points $q\in\partial D$ that minimize the length of the broken paths connecting p to q and q to p.

(iii) the tangent planes of ∂D at all the first reflection points.

It is also shown that, under the *admissibility* condition for D, one can extract the Gauss curvature at an arbitrary first reflection point and the mean curvature with an additional term which depends on the positions of p, p' and the first reflection point. As a byproduct, for an example, a constructive proof of a uniqueness theorem for a spherical obstacle using the bistatic data is also given.

We think that the problem taken up in this paper is a prototype of other various interesting problems. It is quite interesting whether the approach presented here can be applied to them or to develope its necessary modification. Here we mention some of them.

• Consider the sound-hard obstacle case or the obstacles with a dissipative boundary condition (cf. [21]). And also it is quite important to consider the corresponding problem for the *Maxwell system*. These remain open.

• It would be interesting to consider also the case when obstacles are embedded in one of the *two layers* with known different propagation speeds and both the source and receivers are placed in another layer.

• Maybe the most interesting problem is that of extracting geometrical information about an unknown obstacle D *behind* a known obstacle D_0 from the monostatic or bistatic data over a finite time interval. B and B' satisfy $[\overline{B}\cup\overline{B'}]\cap(\overline{D}_0\cup\overline{D})=\emptyset$ and D is optically invisible from B and B' because of the existence of D_0.

• It would be interesting to consider the case when B' is placed in the *shadow region* of D with respect to B. It means that B' is *invisible* from B because of the existence of D between them. This is the case when $[\overline{B}\cup\overline{B'}]\cap\partial D\neq\emptyset$. What information about D can one extract from u_f on $B'\times\,]0,\,T[$?

Although the author's interest is pursuit of the possibility of the enclosure method itself, research by other approaches to the problem taken up in this paper is also expected. Someone may think about the use of *geometrical optics* in the *time domain* as Majda has done in [19] for the problem considered in [18]. See also pages 440-447 in [25] for geometrical optics in the time domain and [22] for Majda's approach. His approach heavily depends on the hyperbolic nature of the governing equation in contrast to our

approach and it should be noted that the existing results by our approach can cover inverse problems for different type of equations like elliptic, parabolic or hyperbolic ones. Since this paper has not aimed at the comparative study of various approaches, we leave it to other opportunities.

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7 Appendix

7.1 Proof of (1.7)

First we prove that

$$\min_{x\in\partial D}\phi(x;p,p')-(\eta+\eta')\ge\min_{x\in\partial D,\,y\in\partial B,\,y'\in\partial B'}\phi(x;y,y').$$

Choose $q\in\partial D$ such that

$$\phi(q;p,p')=\min_{x\in\partial D}\phi(x;p,p').$$

One can find $y_0\in\partial B\cap[q,p]$ and $y_0'\in\partial B'\cap[q,p']$. Since $|y_0-q|=|p-q|-\eta$ and $|y_0'-q|=|p'-q|-\eta'$, we have

$$\phi(q;y_0,y_0')=\phi(q;p,p')-(\eta+\eta').$$

This yields

$$\phi(q;p,p')-(\eta+\eta')\ge\min_{x\in\partial D,\,y\in\partial B,\,y'\in\partial B'}\phi(x;y,y').$$

Next we prove that

$$\min_{x\in\partial D}\phi(x;p,p')-(\eta+\eta')\le\min_{x\in\partial D,\,y\in\partial B,\,y'\in\partial B'}\phi(x;y,y').$$

Choose $q\in\partial D$, $y_0\in\partial B$ and $y_0'\in\partial B'$ such that

$$\phi(q;y_0,y_0')=\min_{x\in\partial D,\,y\in\partial B,\,y'\in\partial B'}\phi(x;y,y').$$

Let $y(s)$ be an arbitrary curve on ∂B such that $y(0)=y_0$. Since $\phi(q;y(s),y_0')$ takes its minimum value at $s=0$, we have

$$\frac{d}{ds}\phi(q;y(s),y_0')|_{s=0}=0.$$

This gives

$$\frac{q-y_0}{|q-y_0|}\cdot\frac{dy}{ds}(0)=0.$$

Since $dy/ds(0)$ can be an arbitrary vector perpendicular to the normal vector at $y_0 \in \partial B$ and q is outside of B, we have

$$\frac{q - y_0}{|q - y_0|} = \frac{y_0 - p}{|y_0 - p|}.$$

This yields $y_0 \in [q, p]$ and similarly we have $y_0' \in [q, p']$. Thus we have $|y_0 - q| = |p - q| - \eta$ and $|y_0' - q| = |p' - q| - \eta'$. This yields

$$\phi(q; y_0, y_0') = \phi(q; p, p') - (\eta + \eta')$$

and thus

$$\phi(q; y_0, y_0') \geq \min_{x \in \partial D} \phi(x; p, p') - (\eta + \eta').$$

7.2 Proof of Proposition 1.1.

Proof of (i). Since every point on ∂D is a point of support of D, we have (3.6). And from (3.6) we have $\epsilon_f^0(y^r) \geq -v_f(y)$ for $d_{\partial D}(y) << 1$ and $y \in D$. Since $v_f \geq 0$ and $\epsilon_f^0 < 0$ (maximum principle), we have, for $d_{\partial D}(y) << 1$ and $y \in D$, $|\epsilon_f^0(y^r)| \leq v_f(y)$. This yields the desired estimate.

Let $c = \min_{x \in \partial D} \phi(x; p, p')$. Assume that there exist two distinct points q and q' on $\Lambda_{\partial D}(p, p')$. Since D is convex, $\overline{D}$ becomes also convex. Thus every point on $[q, q']$ are in $\overline{D}$. Choose an arbitrary point q'' on $[q, q'] \setminus \{q, q'\}$. We have $q'' \in \overline{D}$. Since both q and q' are on $E_c(p, p')$ with $c = \min_{x \in \partial D} \phi(x; p, p')$, it is clear that every point y on $[q, q'] \setminus \{q, q'\}$ satisfies $\phi(y; p, p') < c$ and thus $\phi(q''; p, p') < c$. Since $\phi(x; p, p') \geq c$ for all $x \in \partial D$ and $q'' \in D \cup \partial D$, it must hold that $q'' \in D$. Since p in $\mathbf{R}^3 \setminus \overline{D}$, there exists a point q''' on $[q'', p]$ such that $q''' \in \partial D$. Then it is clear to have $\phi(q'''; p, p') \leq \phi(q''; p, p')$ and thus $\phi(q'''; p, p') < c$. However, since $q''' \in \partial D$, we have $\phi(q'''; p, p') \geq c$. This is a contradiction.
Proof of (ii). Since ∂D is in the half space $(x - q) \cdot \nu_q \leq 0$, it is easy to see that $S_q(\partial D) \leq 0$ as the quadratic form on $T_q(\partial D)$. On the other hand, we have $S_q(E_c(p, p'))$ is positive definite as the quadratic form on $T_q(E_c(p, p')) = T_q(\partial D)$ (see Appendix). Thus $S_q(E_c(p, p')) - S_q(\partial D)$ is positive definite as the quadratic form on the common tangent space and this yields (1.14).

7.3 The shape operator for a spheroid

Since $c > |p - p'|$, we have $[p, p'] \cap E_c(p, p') = \emptyset$. Thus, for all $x \in E_c(p, p')$,

$$\frac{x - p}{|x - p|} + \frac{x - p'}{|x - p'|} \neq 0$$

and hence

$$1 + \frac{x - p}{|x - p|} \cdot \frac{x - p'}{|x - p'|} > 0.$$

Since

$$\nabla \phi(x; p, p') = \frac{x - p}{|x - p|} + \frac{x - p'}{|x - p'|},$$

we conclude that $E_c(p,p')$ is a C^∞ surface and clearly compact. Let ν_x denote the unit *inward normal* for $x \in E_c(p,p')$. We have

$$\nu_x = -\frac{\nabla\phi}{|\nabla\phi|}$$

and note that

$$|\nabla\phi| = \sqrt{2}\sqrt{1 + \frac{x-p}{|x-p|}\cdot\frac{x-p'}{|x-p'|}}.$$

These give the expression

$$-\nu_x = \frac{\boldsymbol{A}(x) + \boldsymbol{A}'(x)}{\sqrt{2(1 + \boldsymbol{A}(x)\cdot\boldsymbol{A}'(x))}}, \tag{A.1}$$

where

$$\boldsymbol{A}(x) = \frac{x-p}{|x-p|},\ \ \boldsymbol{A}'(x) = \frac{x-p'}{|x-p'|}.$$

Now we are ready to prove the following proposition.

Proposition A.1. *Let S_x denote the shape operator at $x \in E_c(p,p')$ with respect to ν_x which is the unit inward normal to $E_c(p,p')$. We have, for all $\boldsymbol{v} \in T_xE_c(p,p')$,*

$$S_x(\boldsymbol{v}) = \frac{\lambda(x)}{\sqrt{2(1 + \boldsymbol{A}(x)\cdot\boldsymbol{A}'(x))}}\left(I_3 - \frac{1}{2}\boldsymbol{A}(x)\otimes\boldsymbol{A}(x) - \frac{1}{2}\boldsymbol{A}'(x)\otimes\boldsymbol{A}'(x)\right)\boldsymbol{v}, \tag{A.2}$$

where

$$\lambda(x) = \frac{1}{|x-p|} + \frac{1}{|x-p'|}.$$

Proof. Let $x = x(\sigma_1,\sigma_2)$ be an equation for $E_c(p,p')$ around $q \in E_c(p,p')$. It means that $x(0,0) = q$ and $\phi(x(\sigma_1,\sigma_2);p,p') = c$. Since

$$\frac{\partial}{\partial\sigma_j}|x-p| = \frac{x-p}{|x-p|}\cdot\frac{\partial x}{\partial\sigma_j},$$

we have

$$\begin{aligned}\frac{\partial}{\partial\sigma_j}\boldsymbol{A}(x) &= \frac{1}{|x-p|}\frac{\partial x}{\partial\sigma_j} - \frac{x-p}{|x-p|^2}\frac{x-p}{|x-p|}\cdot\frac{\partial x}{\partial\sigma_j}\\ &= \frac{1}{|x-p|}\left\{\frac{\partial x}{\partial\sigma_j} - \boldsymbol{A}(x)\left(\boldsymbol{A}(x)\cdot\frac{\partial x}{\partial\sigma_j}\right)\right\}\\ &= \frac{1}{|x-p|}\left(I_3 - \boldsymbol{A}(x)\otimes\boldsymbol{A}(x)\right)\frac{\partial x}{\partial\sigma_j}.\end{aligned}$$

This together with a corresponding expression for $(\partial/\partial\sigma_j)\boldsymbol{A}'(x)$ gives

$$\frac{\partial}{\partial\sigma_j}(\boldsymbol{A}(x) + \boldsymbol{A}'(x)) = \left(\lambda(x)I_3 - \frac{1}{|x-p|}\boldsymbol{A}(x)\otimes\boldsymbol{A}(x) - \frac{1}{|x-p'|}\boldsymbol{A}'(x)\otimes\boldsymbol{A}'(x)\right)\frac{\partial x}{\partial\sigma_j}.$$

Moreover, since

$$\frac{\partial x}{\partial \sigma_j} \cdot \boldsymbol{A}(x) = -\frac{\partial x}{\partial \sigma_j} \cdot \boldsymbol{A}'(x),$$

we have

$$\frac{\partial}{\partial \sigma_j}(\boldsymbol{A}(x) \cdot \boldsymbol{A}'(x))$$

$$= \frac{1}{|x-p|} \left(I_3 - \boldsymbol{A}(x) \otimes \boldsymbol{A}(x)\right) \frac{\partial x}{\partial \sigma_j} \cdot \boldsymbol{A}'(x) + \frac{1}{|x-p'|} \boldsymbol{A}(x) \cdot \left(I_3 - \boldsymbol{A}'(x) \otimes \boldsymbol{A}'(x)\right) \frac{\partial x}{\partial \sigma_j}$$

$$= \frac{1}{|x-p|} \frac{\partial x}{\partial \sigma_j} \cdot \boldsymbol{A}'(x) + \frac{1}{|x-p'|} \frac{\partial x}{\partial \sigma_j} \cdot \boldsymbol{A}(x)$$

$$-\frac{1}{|x-p|} \left(\boldsymbol{A}(x) \cdot \frac{\partial x}{\partial \sigma_j}\right) (\boldsymbol{A}(x) \cdot \boldsymbol{A}'(x)) - \frac{1}{|x-p'|} \left(\boldsymbol{A}'(x) \cdot \frac{\partial x}{\partial \sigma_j}\right) (\boldsymbol{A}(x) \cdot \boldsymbol{A}'(x))$$

$$= -\frac{1}{|x-p|}(1 + \boldsymbol{A}(x) \cdot \boldsymbol{A}'(x)) \left(\boldsymbol{A}(x) \cdot \frac{\partial x}{\partial \sigma_j}\right) - \frac{1}{|x-p'|}(1 + \boldsymbol{A}(x) \cdot \boldsymbol{A}'(x)) \left(\boldsymbol{A}'(x) \cdot \frac{\partial x}{\partial \sigma_j}\right)$$

$$= -(1 + \boldsymbol{A}(x) \cdot \boldsymbol{A}'(x)) \left(\frac{1}{|x-p|} \boldsymbol{A}(x) \cdot \frac{\partial x}{\partial \sigma_j} + \frac{1}{|x-p'|} \boldsymbol{A}'(x) \cdot \frac{\partial x}{\partial \sigma_j}\right)$$

we obtain

$$\frac{\partial}{\partial \sigma_j} \frac{1}{\sqrt{1 + \boldsymbol{A}(x) \cdot \boldsymbol{A}'(x)}}$$

$$= -\frac{1}{2} \frac{\frac{\partial}{\partial \sigma_j}(\boldsymbol{A}(x) \cdot \boldsymbol{A}'(x))}{(1 + \boldsymbol{A}(x) \cdot \boldsymbol{A}'(x))^{3/2}}$$

$$= \frac{1}{2(1 + \boldsymbol{A}(x) \cdot \boldsymbol{A}'(x))^{1/2}} \left(\frac{1}{|x-p|} \boldsymbol{A}(x) \cdot \frac{\partial x}{\partial \sigma_j} + \frac{1}{|x-p'|} \boldsymbol{A}'(x) \cdot \frac{\partial x}{\partial \sigma_j}\right).$$

Thus one gets

$$\frac{\partial}{\partial \sigma_j} \frac{\boldsymbol{A}(x) + \boldsymbol{A}'(x)}{\sqrt{1 + \boldsymbol{A}(x) \cdot \boldsymbol{A}'(x)}} = \frac{\boldsymbol{A}(x) + \boldsymbol{A}'(x)}{2\sqrt{1 + \boldsymbol{A}(x) \cdot \boldsymbol{A}'(x)}} \left(\frac{1}{|x-p|} \frac{\partial x}{\partial \sigma_j} \cdot \boldsymbol{A}(x) + \frac{1}{|x-p'|} \frac{\partial x}{\partial \sigma_j} \cdot \boldsymbol{A}'(x)\right)$$

$$+\frac{1}{\sqrt{1 + \boldsymbol{A}(x) \cdot \boldsymbol{A}'(x)}} \left(\lambda(x) I_3 - \frac{1}{|x-p|} \boldsymbol{A}(x) \otimes \boldsymbol{A}(x) - \frac{1}{|x-p'|} \boldsymbol{A}'(x) \otimes \boldsymbol{A}'(x)\right) \frac{\partial x}{\partial \sigma_j}$$

and thus

$$\sqrt{1+\boldsymbol{A}(x)\cdot\boldsymbol{A}'(x)}\frac{\partial}{\partial\sigma_j}\frac{\boldsymbol{A}(x)+\boldsymbol{A}'(x)}{\sqrt{1+\boldsymbol{A}(x)\cdot\boldsymbol{A}'(x)}}$$

$$=\frac{1}{2}(\boldsymbol{A}(x)+\boldsymbol{A}'(x))\frac{1}{|x-p|}\frac{\partial x}{\partial\sigma_j}\cdot\boldsymbol{A}(x)+\frac{1}{2}(\boldsymbol{A}(x)+\boldsymbol{A}'(x))\frac{1}{|x-p'|}\frac{\partial x}{\partial\sigma_j}\cdot\boldsymbol{A}'(x)$$

$$-\boldsymbol{A}(x)\frac{1}{|x-p|}\frac{\partial x}{\partial\sigma_j}\cdot\boldsymbol{A}(x)-\boldsymbol{A}'(x)\frac{1}{|x-p'|}\frac{\partial x}{\partial\sigma_j}\cdot\boldsymbol{A}'(x)+\lambda(x)\frac{\partial x}{\partial\sigma_j}$$

$$=-\frac{1}{2}\boldsymbol{A}(x)\frac{1}{|x-p|}\frac{\partial x}{\partial\sigma_j}\cdot\boldsymbol{A}(x)-\frac{1}{2}\boldsymbol{A}'(x)\frac{1}{|x-p'|}\frac{\partial x}{\partial\sigma_j}\cdot\boldsymbol{A}'(x)$$

$$+\frac{1}{2}\boldsymbol{A}'(x)\frac{1}{|x-p|}\frac{\partial x}{\partial\sigma_j}\cdot\boldsymbol{A}(x)+\frac{1}{2}\boldsymbol{A}(x)\frac{1}{|x-p'|}\frac{\partial x}{\partial\sigma_j}\cdot\boldsymbol{A}'(x)+\lambda(x)\frac{\partial x}{\partial\sigma_j}$$

$$=-\frac{1}{2}\boldsymbol{A}(x)\frac{1}{|x-p|}\frac{\partial x}{\partial\sigma_j}\cdot\boldsymbol{A}(x)-\frac{1}{2}\boldsymbol{A}'(x)\frac{1}{|x-p'|}\frac{\partial x}{\partial\sigma_j}\cdot\boldsymbol{A}'(x)$$

$$-\frac{1}{2}\boldsymbol{A}'(x)\frac{1}{|x-p|}\frac{\partial x}{\partial\sigma_j}\cdot\boldsymbol{A}'(x)-\frac{1}{2}\boldsymbol{A}(x)\frac{1}{|x-p'|}\frac{\partial x}{\partial\sigma_j}\cdot\boldsymbol{A}(x)+\lambda(x)\frac{\partial x}{\partial\sigma_j}$$

$$=-\frac{1}{2}\boldsymbol{A}(x)\lambda(x)\frac{\partial x}{\partial\sigma_j}\cdot\boldsymbol{A}(x)-\frac{1}{2}\boldsymbol{A}'(x)\lambda(x)\frac{\partial x}{\partial\sigma_j}\cdot\boldsymbol{A}'(x)+\lambda(x)\frac{\partial x}{\partial\sigma_j}.$$

Therefore we obtain

$$\frac{\partial}{\partial\sigma_j}\frac{\boldsymbol{A}(x)+\boldsymbol{A}'(x)}{\sqrt{1+\boldsymbol{A}(x)\cdot\boldsymbol{A}'(x)}}$$

$$=\frac{\lambda(x)}{\sqrt{1+\boldsymbol{A}(x)\cdot\boldsymbol{A}'(x)}}\left(I_3-\frac{1}{2}\boldsymbol{A}(x)\otimes\boldsymbol{A}(x)-\frac{1}{2}\boldsymbol{A}'(x)\otimes\boldsymbol{A}'(x)\right)\frac{\partial x}{\partial\sigma_j}$$

and hence

$$-\frac{\partial}{\partial\sigma_j}\nu_x=\frac{\lambda(x)}{\sqrt{2(1+\boldsymbol{A}(x)\cdot\boldsymbol{A}'(x))}}\left(I_3-\frac{1}{2}\boldsymbol{A}(x)\otimes\boldsymbol{A}(x)-\frac{1}{2}\boldsymbol{A}'(x)\otimes\boldsymbol{A}'(x)\right)\frac{\partial x}{\partial\sigma_j}.$$

Since

$$S_x\left(\xi_1\frac{\partial x}{\partial\sigma_1}|_{\sigma=0}+\xi_2\frac{\partial x}{\partial\sigma_2}|_{\sigma=0}\right)=-\left(\xi_1\frac{\partial}{\partial\sigma_1}\nu_x|_{\sigma=0}+\xi_2\frac{\partial}{\partial\sigma_2}\nu_x|_{\sigma=0}\right),$$

we obtain (A.2).

□

From (A.2) one can compute the principle curvatures at $x\in E_c(p,p')$.

Consider the case $\boldsymbol{A}(x)\neq\boldsymbol{A}'(x)$. Since $\boldsymbol{A}(x)$ and $\boldsymbol{A}'(x)$ are unit vectors and $\boldsymbol{A}(x)\neq-\boldsymbol{A}'(x)$, we have $\boldsymbol{v}=\boldsymbol{A}(x)\times\boldsymbol{A}'(x)\neq 0$. Since $\boldsymbol{A}(x)\cdot\boldsymbol{v}=\boldsymbol{A}'(x)\cdot\boldsymbol{v}=0$, $\boldsymbol{v}$ satisfies

$$S_x(\boldsymbol{v})=\frac{\lambda(x)}{\sqrt{2(1+\boldsymbol{A}(x)\cdot\boldsymbol{A}'(x))}}\boldsymbol{v}.$$

Next choose $\boldsymbol{v}' = \boldsymbol{A}(x) - \boldsymbol{A}'(x)$. Since ν_x and $\boldsymbol{A}(x) + \boldsymbol{A}'(x)$ are parallel, $\boldsymbol{v}' \in T_x E_c(p, p')$. We have

$$S_x(\boldsymbol{v}') = \frac{\lambda(x)\sqrt{1 + \boldsymbol{A}(x) \cdot \boldsymbol{A}'(x)}}{2\sqrt{2}} \boldsymbol{v}'.$$

Note also that $\boldsymbol{v}$ and $\boldsymbol{v}'$ are perpendicular to each other. Therefore the eigenvalues of S_x consists of two real numbers:

$$\begin{aligned} k_1(x) &= \frac{\lambda(x)}{\sqrt{2(1 + \boldsymbol{A}(x) \cdot \boldsymbol{A}'(x))}}, \\ k_2(x) &= \frac{\lambda(x)\sqrt{1 + \boldsymbol{A}(x) \cdot \boldsymbol{A}'(x)}}{2\sqrt{2}}. \end{aligned} \tag{A.3}$$

If $\boldsymbol{A}(x) = \boldsymbol{A}'(x)$, then $\nu_x = \boldsymbol{A}(x)$. Since $\boldsymbol{v} \cdot \nu_x = 0$ for all $\boldsymbol{v} \in T_x(E_c(p, p'))$, from (A.2) we obtain

$$S_x(\boldsymbol{v}) = \frac{\lambda(x)}{2} \boldsymbol{v}, \ \forall \boldsymbol{v} \in \ T_x(E_c(p, p')).$$

Thus the set of all eigenvalues of S_x consists of only $\lambda(x)/2$. Therefore $k_1(x)$, $k_2(x)$ given by (A.3) covers also this special case. Note that $k_2(x) \le k_1(x)$ and $k_1(x) = k_2(x)$ if and only if $\boldsymbol{A}(x) = \boldsymbol{A}'(x)$. Therefore the Gauss curvature $K(x)$ at $x \in E_c(p, p')$ and the mean curvature $H(x)$ with respect to ν_x are

$$K(x) = k_1(x) k_2(x) = \frac{\lambda(x)^2}{4},$$

$$H(x) = \frac{k_1(x) + k_2(x)}{2} = \frac{\lambda(x)}{8}(3 + \boldsymbol{A}(x) \cdot \boldsymbol{A}'(x)).$$

7.4 Proof of Lemma 5.1

Replacing p' with $p' + s\boldsymbol{A}'$, it follows from (4.18) and (4.19) that

$$\begin{aligned} &\det\,(S_q(E_{c-s}(p, p' + s\boldsymbol{A}')) - S_q(\partial D)) = \det\left(\frac{\lambda(q; p, p' + s\boldsymbol{A}')}{\sqrt{2(1 + \boldsymbol{A} \cdot \boldsymbol{A}')}} \delta_{kj} - M_{kj} \right) \\ &= \left(\frac{\lambda(q; p, p' + s\boldsymbol{A}')}{\sqrt{2(1 + \boldsymbol{A} \cdot \boldsymbol{A}')}} \right)^2 - \frac{\lambda(q; p, p' + s\boldsymbol{A}')}{\sqrt{2(1 + \boldsymbol{A} \cdot \boldsymbol{A}')}} \mathrm{Trace}\,(M_{kj}) + \det\,(M_{kj}), \end{aligned} \tag{A.4}$$

where

$$M_{kj} = \frac{\lambda(q; p, p' + s\boldsymbol{A}')}{2\sqrt{2(1 + \boldsymbol{A} \cdot \boldsymbol{A}')}} (\boldsymbol{A} \cdot \boldsymbol{e}_k \boldsymbol{A} \cdot \boldsymbol{e}_j + \boldsymbol{A}' \cdot \boldsymbol{e}_k \boldsymbol{A}' \cdot \boldsymbol{e}_j) + \frac{\partial^2 f}{\partial \sigma_k \sigma_j}(0). \tag{A.5}$$

From (4.14) we have

$$\boldsymbol{A} \cdot \nu_q = \boldsymbol{A}' \cdot \nu_q = -\sqrt{\frac{1 + \boldsymbol{A} \cdot \boldsymbol{A}'}{2}}.$$

These give

$$\sum_{k=1}^{2}(|\boldsymbol{A}\cdot\boldsymbol{e}_k|^2+|\boldsymbol{A}'\cdot\boldsymbol{e}_k|^2)=1-\boldsymbol{A}\cdot\boldsymbol{A}'.$$

Thus one gets

$$\text{Trace}\,(M_{kj})=\frac{\lambda(q;p,p'+s\boldsymbol{A}')(1-\boldsymbol{A}\cdot\boldsymbol{A}')}{2\sqrt{2(1+\boldsymbol{A}\cdot\boldsymbol{A}')}}+2H_{\partial D}(q). \tag{A.6}$$

For the computation of $\det(M_{kj})$ we prepare the following formula.

Proposition A.2. *Let B be a 2×2-matrix, $\boldsymbol{c}$ and $\boldsymbol{c}'$ be two-dimensional vectors. Let γ be a constant. Let*

$$M=\gamma(\boldsymbol{c}\otimes\boldsymbol{c}+\boldsymbol{c}'\otimes\boldsymbol{c}')+B. \tag{A.7}$$

We have

$$det\,M$$

$$=\gamma^2\left(det\begin{pmatrix}c_1 & c_1'\\ c_2 & c_2'\end{pmatrix}\right)^2+\gamma\left(B\begin{pmatrix}c_2\\ -c_1\end{pmatrix}\cdot\begin{pmatrix}c_2\\ -c_1\end{pmatrix}+B\begin{pmatrix}c_2'\\ -c_1'\end{pmatrix}\cdot\begin{pmatrix}c_2'\\ -c_1'\end{pmatrix}\right)+det\,B.$$

Proof. We have

$$\det M$$

$$=(\gamma(c_1^2+c_1'^2)+b_{11})(\gamma(c_2^2+c_2'^2)+b_{22})-(\gamma(c_1c_2+c_1'c_2')+b_{12})(\gamma(c_2c_1+c_2'c_1')+b_{21})$$

$$=\gamma^2(c_1^2+c_1'^2)(c_2^2+c_2'^2)+\gamma((c_1^2+c_1'^2)b_{22}+(c_2^2+c_2'^2)b_{11})+b_{11}b_{22}$$

$$-\gamma^2(c_1c_2+c_1'c_2')(c_2c_1+c_2'c_1')-\gamma((c_1c_2+c_1'c_2')b_{21}+(c_2c_1+c_2'c_1')b_{12})-b_{12}b_{21}$$

$$=\gamma^2((c_1^2+c_1'^2)(c_2^2+c_2'^2)-(c_1c_2+c_1'c_2')^2)$$

$$+\gamma((c_1^2+c_1'^2)b_{22}+(c_2^2+c_2'^2)b_{11}-(c_1c_2+c_1'c_2')b_{21}-(c_2c_1+c_2'c_1')b_{12})+\det B$$

$$=\gamma^2(c_1c_2'-c_1'c_2)^2$$

$$+\gamma((c_1^2+c_1'^2)b_{22}+(c_2^2+c_2'^2)b_{11}-(c_1c_2+c_1'c_2')(b_{12}+b_{21}))+\det B.$$

□

Note that (M_{kj}) given by (A.5) coincides with (A.7) in the case when $\boldsymbol{c}=(\boldsymbol{A}\cdot\boldsymbol{e}_j)$, $\boldsymbol{c}'=(\boldsymbol{A}'\cdot\boldsymbol{e}_j)$, $B=\nabla^2 f(0)$ and

$$\gamma=\frac{\lambda(q;p,p'+s\boldsymbol{A}')}{2\sqrt{2(1+\boldsymbol{A}\cdot\boldsymbol{A}')}}.$$

From (4.15) we have

$$\det \begin{pmatrix} \boldsymbol{A}\cdot\boldsymbol{e}_1 & \boldsymbol{A}'\cdot\boldsymbol{e}_1 \\ \boldsymbol{A}\cdot\boldsymbol{e}_2 & \boldsymbol{A}'\cdot\boldsymbol{e}_2 \end{pmatrix} = \det \begin{pmatrix} \boldsymbol{A}\cdot\boldsymbol{e}_1 & -\boldsymbol{A}\cdot\boldsymbol{e}_1 \\ \boldsymbol{A}\cdot\boldsymbol{e}_2 & -\boldsymbol{A}\cdot\boldsymbol{e}_2 \end{pmatrix} = 0; \tag{A.8}$$

from (4.19) we have

$$\begin{aligned} &\nabla^2 f(0) \begin{pmatrix} \boldsymbol{A}\cdot\boldsymbol{e}_2 \\ -\boldsymbol{A}\cdot\boldsymbol{e}_1 \end{pmatrix} \cdot \begin{pmatrix} \boldsymbol{A}\cdot\boldsymbol{e}_2 \\ -\boldsymbol{A}\cdot\boldsymbol{e}_1 \end{pmatrix} \\ &= S_q(\partial D)((\boldsymbol{A}\cdot\boldsymbol{e}_2)\boldsymbol{e}_1 - (\boldsymbol{A}\cdot\boldsymbol{e}_1)\boldsymbol{e}_2)\cdot((\boldsymbol{A}\cdot\boldsymbol{e}_2)\boldsymbol{e}_1 - (\boldsymbol{A}\cdot\boldsymbol{e}_1)\boldsymbol{e}_2). \end{aligned} \tag{A.9}$$

By Lemma 4.3, we have

$$\boldsymbol{A}\times\boldsymbol{A}' = -\sqrt{2(1+\boldsymbol{A}\cdot\boldsymbol{A}')}((\boldsymbol{A}\cdot\boldsymbol{e}_2)\boldsymbol{e}_1 - (\boldsymbol{A}\cdot\boldsymbol{e}_1)\boldsymbol{e}_2).$$

This gives

$$\begin{aligned} &S_q(\partial D)((\boldsymbol{A}\cdot\boldsymbol{e}_2)\boldsymbol{e}_1 - (\boldsymbol{A}\cdot\boldsymbol{e}_1)\boldsymbol{e}_2)\cdot((\boldsymbol{A}\cdot\boldsymbol{e}_2)\boldsymbol{e}_1 - (\boldsymbol{A}\cdot\boldsymbol{e}_1)\boldsymbol{e}_2) \\ &= \frac{S_q(\partial D)(\boldsymbol{A}\times\boldsymbol{A}')\cdot(\boldsymbol{A}\times\boldsymbol{A}')}{2(1+\boldsymbol{A}\cdot\boldsymbol{A}')} \end{aligned}$$

and thus from (A.9) one gets

$$\nabla^2 f(0) \begin{pmatrix} \boldsymbol{A}\cdot\boldsymbol{e}_2 \\ -\boldsymbol{A}\cdot\boldsymbol{e}_1 \end{pmatrix} \cdot \begin{pmatrix} \boldsymbol{A}\cdot\boldsymbol{e}_2 \\ -\boldsymbol{A}\cdot\boldsymbol{e}_1 \end{pmatrix} = \frac{S_q(\partial D)(\boldsymbol{A}\times\boldsymbol{A}')\cdot(\boldsymbol{A}\times\boldsymbol{A}')}{2(1+\boldsymbol{A}\cdot\boldsymbol{A}')}. \tag{A.10}$$

Since

$$(\boldsymbol{A}'\cdot\boldsymbol{e}_2)\boldsymbol{e}_1 - (\boldsymbol{A}'\cdot\boldsymbol{e}_1)\boldsymbol{e}_2 = -((\boldsymbol{A}\cdot\boldsymbol{e}_2)\boldsymbol{e}_1 - (\boldsymbol{A}\cdot\boldsymbol{e}_1)\boldsymbol{e}_2),$$

we obtain also

$$\nabla^2 f(0) \begin{pmatrix} \boldsymbol{A}'\cdot\boldsymbol{e}_2 \\ -\boldsymbol{A}'\cdot\boldsymbol{e}_1 \end{pmatrix} \cdot \begin{pmatrix} \boldsymbol{A}'\cdot\boldsymbol{e}_2 \\ -\boldsymbol{A}'\cdot\boldsymbol{e}_1 \end{pmatrix} = \frac{S_q(\partial D)(\boldsymbol{A}\times\boldsymbol{A}')\cdot(\boldsymbol{A}\times\boldsymbol{A}')}{2(1+\boldsymbol{A}\cdot\boldsymbol{A}')}. \tag{A.11}$$

From Proposition A.2, (A.8), (A.10) and (A.11) we obtain

$$\det(M_{kj}) = \frac{\lambda(q;p,p'+s\boldsymbol{A}')}{\sqrt{2}(1+\boldsymbol{A}\cdot\boldsymbol{A}')^{3/2}} S_q(\partial D)(\boldsymbol{A}\times\boldsymbol{A}')\cdot(\boldsymbol{A}\times\boldsymbol{A}') + K_{\partial D}(q).$$

Substituting this together with (A.6) into (A.4), we obtain (5.6).

References

[1] Athanasiadis, C., Martin, P. A. and Stratis, I. G., On spherical-wave scattering by a spherical scatterer and related near-field inverse problems, IMA J. Appl. Math., **66**(2001), 539-549.

[2] Bleistein, N. and Handelsman, R. A., Asymptotic expansions of integrals, Dover Publications, New York, 1986.

[3] Colton, D. and Kress, R., Inverse acoustic and electromagnetic scattering theory, Springer, second edition, 1998.

[4] Cox, H., Fundamentals of bistatic active sonar, in Underwater acoustic data processing, Chan Y. T. (Ed.), 1989, 3-24. Kluwer Academic Publishers.

[5] Dautray, R. and Lions, J-L., Mathematical analysis and numerical methods for sciences and technology, Evolution problems I, Vol. **5**, Springer-Verlag, Berlin, 1992.

[6] Gilbarg, D. and Trudinger, N. S., Elliptic partial differential equations of second order, second.ed., Springer-Verlag, Berlin, Heidelberg, New York,Tokyo, 1983.

[7] Ikehata, M., Enclosing a polygonal cavity in a two-dimensional bounded domain from Cauchy data, Inverse Problems, **15**(1999), 1231-1241.

[8] Ikehata, M., Extracting discontinuity in a heat conductive body. One-space dimensional case, Applicable Analysis, **86**(2007), no. 8, 963-1005.

[9] Ikehata, M., The enclosure method for inverse obstacle scattering problems with dynamical data over a finite time interval, Inverse Problems, **26**(2010) 055010(20pp).

[10] Ikehata, M., The framework of the enclosure method with dynamical data and its applications, Inverse Problems, **27**(2011) 065005(16pp).

[11] Ikehata, M., The enclosure method for inverse obstacle scattering problems with dynamical data over a finite time interval: II. Obstacles with a dissipative boundary or finite refractive index and back-scattering data, Inverse Problems, **28**(2012) 045010(29pp).

[12] Ikehata, M., An inverse acoustic scattering problem inside a cavity with dynamical back-scattering data, Inverse Problems, **28**(2012) 095016(24pp).

[13] Ikehata, M. and Itou, H., On reconstruction of a cavity in a linearized viscoelastic body from infinitely many transient boundary data, Inverse Problems, **28**(2012) 125003(19pp).

[14] Ikehata, M. and Kawashita, M., The enclosure method for the heat equation, Inverse Problems, **25**(2009) 075005(10pp).

[15] Ikehata, M. and Kawashita, M., On the reconstruction of inclusions in a heat conductive body from dynamical boundary data over a finite time interval, Inverse Problems, **26**(2010) 095004(15pp).

[16] Isakov, V., On uniqueness for a discontinuity surface of the speed of propagation, J. Inv. Ill-Posed Problems, **4**(1996), 33-38.

[17] Isakov, V., Inverse problems for partial differential equations (Second Edition), Springer, New York, 2006.

[18] Lax, P. D. and Phillips, R. S., The scattering of sound waves by an obstacle, Comm. Pure Appl. Math., **30**(1977), 195-233.

[19] Majda, A., A representation formula for the scattering operator and the inverse problem for arbitrary bodies, Comm. Pure and Appl. Math., **30**(1977), 165-194.

[20] Mizohata, S., Theory of partial differential equations, Cambridge Univ. Press, Cambridge, 1973.

[21] Propst, G. and Prüss, J., On wave equations with boundary dissipation of memory type, Integral Equations Appl., **8**(1996), no.1, 99-123.

[22] Petkov, V. and Stoyanov, L., Sojourn times, singularities of the scattering kernel and inverse problems, Inside Out: Inverse Problems, MSRI Publications, **47**, 2003, 297-332.

[23] Rakesh, An inverse impedance transmission problem for the wave equation, Comm. in partial differential equations, **18**(1993), 583-600.

[24] Rakesh, Some results on inverse obstacle problems for the wave equation, Algebra i Analiz, **8**(1996), no.2, 157-161.

[25] Taylor, M. E., Partial differential equations I, Basic theory, Springer, New York, 1997.

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